The Character Table Of A Group Of Shape $(2 \times 2{\cdot}G){:}2$

Richard W. Barraclough

April 10, 2006

Abstract

We use the technique of Fischer matrices to write a program to produce the character table of a group of shape $(2 \times 2{\cdot}G){:}2$ from the character tables of G, $G{:}2$, $2{\cdot}G$ and $2{\cdot}G{:}2$. We also supply a simplified proof of a result frequently used in the Fischer matrices method.

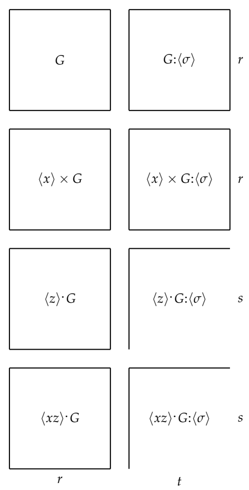
Let G be a finite group with an automorphism of order 2 and a double cover. We may form a group of shape $(2 \times 2{\cdot}G){:}2$ which we write as

$$(\langle x \rangle \times \langle z \rangle{\cdot}G){:}\langle \sigma \rangle$$

so that σ acts to swap x with xz and $\langle x, z, \sigma \rangle \cong D_8$. This group has three interesting subgroups of index 2:

$$G^+ = \langle z \rangle{\cdot}G{:}\langle \sigma \rangle \qquad G^- = \langle z \rangle{\cdot}G{\cdot}\langle x\sigma \rangle \qquad G^0 = \langle x \rangle \times \langle z \rangle{\cdot}G$$

Here G^+ and G^- are representatives of the two isomorphism classes of groups of shape $2{\cdot}G.2$ (they are isoclinic, as explained in [CCN+85]). To produce the character table of $(2 \times 2{\cdot}G){:}2$ all we must do is determine the class fusion from G^0 and the character values on the outer conjugacy classes. The ATLAS [CCN+85] map of the character table of $(2 \times 2{\cdot}G){:}2$ is:



Here G has r conjugacy classes, $2{\cdot}G$ has s characters that are faithful on z, and $G{:}2$ has t conjugacy classes of outer elements.

Let χ be a character in the $\langle z \rangle{\cdot}G$ square, then $\chi(z) = \chi(xz) = -\chi(1)$ and $\chi(x) = \chi(1)$. However, $\chi^\sigma(x) = \chi(xz) = -\chi(1)$ and $\chi^\sigma(xz) = \chi(x) = \chi(1)$. Thus σ must fuse characters in the $\langle z\rangle{\cdot}G$ square with characters in the $\langle xz\rangle{\cdot}G$ square. Therefore then $\langle z\rangle{\cdot}G{:}\langle\sigma\rangle$ and $\langle xz\rangle{\cdot}G{:}\langle\sigma\rangle$ squares contain only zeros. Indeed, there is no subgroup of shape $2{\cdot}G{:}2$.

Each outer class $[g]$ of $G{:}2$ lifts to two classes in $2^2{\cdot}G{:}2$, namely $[g, zg]$ and $[xg, xzg]$. To see this observe that $g^x = xgx = xxzg = zg$, which also shows that if the order of g is indivisible by 4 then the order of xg is twice the order of g. A character $\chi \in \operatorname{Irr} G{:}2$ takes value $\chi(xg) = -\chi(g)$ which, it would seem, completes the table.

We wish to improve the above analysis to sufficient precision to allow the character table to actually be produced. There are about 12 different ways in which conjugacy classes of $G{:}2$ can lift to $2^2{\cdot}G{:}2$, and these are explained in the case analysis beginning on page 4 in section 3. (This number is approximate because there are some subtleties involving the element orders which one may use to produce further cases.) We choose to work with the example $G = \mathrm{Fi}_{22}$ because the character table of $2^2{\cdot}\mathrm{Fi}_{22}{:}2$ is not stored in GAP [GAP02], and most of the conjugacy class types occur (the three that don't can be seen in A_5 or $\mathrm{L}_2(17)$).

We compute the character table using Fischer matrices as in this case we can eliminate all uncertainty about the entries of the matrices. Thus assembling the character table is easy. We shall need the character table of $G{:}2$ and the projective character table of G. This is no more information that we needed above, for the projective character table of G is easily deduced from the character table of $2{\cdot}G$.

1 Fischer Matrices

The technique of Fischer matrices [Fis91] seems generally to have been used to calculate character tables of maximal subgroups of sporadic simple groups and their automorphism groups, and recently has enjoyed something of a revival, *e.g.*, [AM03] and similar papers.

The Fischer matrices method relies on the fact that every irreducible character can be obtained by induction from the inertia groups. Specifically, let $\bar{G} = N.G$ be a group and let $\theta_1, \theta_2, \ldots, \theta_t$ be representatives for the orbits of G on $\operatorname{Irr} N$ (by convention $\theta_1 = 1$). Let $\bar{H}_i$ be the inertia group of θ_i in $\bar{G}$. If ψ_i is a (possibly projective) extension of θ_i to $\bar{H}_i$ then $\psi_i\eta\!\uparrow\!\bar{G}$ is irreducible, where η is inflated from $H_i = \bar{H}_i/N$. In fact

$$\operatorname{Irr}\bar{G} = \bigcup_{i=1}^{t}\left\{ (\psi_i\eta)\!\uparrow\!\bar{G} \mid \eta \in \operatorname{Irr}\bar{H}_i \text{ and } N \subseteq \ker\eta \right\}$$

This is shown, for example, in [Isa94] and [Kar85].

We now let $[g]$ be a conjugacy class of G. $H_i \cap [g]$ splits into a number of H_i conjugacy classes, let representatives be $y_k \in H_i$ for $1 \leqslant k \leqslant r$. Let these lift to classes with representatives y_{l_k} in $\bar{H}_i$. When $\hat{\eta} \in \operatorname{Irr} H_i$ lifts to $\eta \in \operatorname{Irr}\bar{H}_i$ this gives $\eta(y_{l_k}) = \hat{\eta}(y_k)$ for all l.

Theorem 1 *With notation as above,*

$$(\psi_i\eta)\!\uparrow\!\bar{G}(x_j) = \sum_{k=1}^{r}\hat{\eta}(y_k)\sum_{\{l \mid y_{l_k}\sim_{\bar{H}_i}x_j\}}\frac{|C_{\bar{G}}(x_j)|}{|C_{\bar{H}_i}(y_{l_k})|}\psi_i(y_{l_k})$$

Proof. The induction formula for chosen i and j gives

$$(\psi_i\eta)\uparrow\bar{G}(x_j) = \frac{1}{|\bar{H}_i|} \sum_{\{\bar{h}\in\bar{G}|x_j^{\bar{h}}\in\bar{H}_i\}} \psi_i\eta(x_j^{\bar{h}})$$

Now, the classes $[y_{l_k}]_{\bar{H}_i}$ partition $\bar{H}_i$, so the statement $x_j^{\bar{h}} \in \bar{H}_i$ is equivalent to $x_j^{\bar{h}} \sim_{\bar{H}_i} y_{l_k}$ for those y_{l_k} that are $\bar{G}$-conjugate to x_j, *i.e.*,

$$(\psi_i\eta)\uparrow\bar{G}(x_j) = \frac{1}{|\bar{H}_i|} \sum_{k=1}^{r} \sum_{\{l|y_{l_k}\sim_{\bar{G}}x_j\}} \sum_{\{\bar{h}\in\bar{G}|x_j^{\bar{h}}\sim_{\bar{H}_i}y_{l_k}\}} \psi_i\eta(x_j^{\bar{h}})$$

But $\psi_i\eta$ is a class function of $\bar{H}_i$ and so its value is constant over the range of the rightmost sum above. Hence

$$\begin{aligned}
(\psi_i\eta)\uparrow\bar{G}(x_j) &= \frac{1}{|\bar{H}_i|} \sum_{k=1}^{r} \sum_{\{l|y_{l_k}\sim_{\bar{G}}x_j\}} |\{\bar{h}\in\bar{G} \mid x_j^{\bar{h}} \sim_{\bar{H}_i} y_{l_k}\}|\psi_i\eta(y_{l_k}) \\
&= \frac{1}{|\bar{H}_i|} \sum_{k=1}^{r} \sum_{\{l|y_{l_k}\sim_{\bar{G}}x_j\}} |[y_{l_k}]_{\bar{H}_i}||\{\bar{h}\in\bar{G} \mid x_j^{\bar{h}} = y_{l_k}\}|\psi_i\eta(y_{l_k}) \\
&= \frac{1}{|\bar{H}_i|} \sum_{k=1}^{r} \sum_{\{l|y_{l_k}\sim_{\bar{G}}x_j\}} |[y_{l_k}]_{\bar{H}_i}||\{\bar{h}\in\bar{G} \mid x_j^{\bar{h}} = x_j\}|\psi_i\eta(y_{l_k}) \\
&= \sum_{k=1}^{r} \sum_{\{l|y_{l_k}\sim_{\bar{G}}x_j\}} \frac{|C_{\bar{G}}(x_j)|}{|C_{\bar{H}_i}(y_{l_k})|}\psi_i\eta(y_{l_k}) \\
&= \sum_{k=1}^{r} \hat{\eta}(y_k) \sum_{\{l|y_{l_k}\sim_{\bar{G}}x_j\}} \frac{|C_{\bar{G}}(x_j)|}{|C_{\bar{H}_i}(y_{l_k})|}\psi_i(y_{l_k})
\end{aligned}$$

□

Note that by writing

$$a_{kj}^{(i)} = \sum_{\{l|y_{l_k}\sim_{\bar{G}}x_j\}} \frac{|C_{\bar{G}}(x_j)|}{|C_{\bar{H}_i}(y_{l_k})|}\psi_i(y_{l_k}) \qquad (2)$$

(*c.f.* equation 1.10.5 in [Fis91]) Theorem 1 becomes

$$\psi_i\eta\uparrow\bar{G}(x_j) = \sum_{k=1}^{r} a_{kj}^{(i)}\hat{\eta}(y_k)$$

This can be interpreted as multiplication of the matrix $M_i(g) = (a_{kj}^{(i)})$ by a portion of the character table of H_i. The Fischer matrix for the class $[g]_G$ is

$$M(g) = \begin{pmatrix} M_1(g) \\ M_2(g) \\ \vdots \\ M_t(g) \end{pmatrix}$$

where if $H_i \cap [g] = \emptyset$ then $M_i(g)$ is not defined and is omitted from $M(g)$.

The various properties of character tables can be used to deduce constraints on the entries of the Fischer matrices, *e.g.*, the orthogonality relations give a weighted orthogonality for the Fischer matrices. The Fischer matrices method relies on using these properties to deduce the entries of the matrices.

2 The Character Table Of $2^2{\cdot}\mathrm{Fi}_{22}{:}2$

The non-split extension of $N = 2^2$ by $G = \mathrm{Fi}_{22}{:}2$ provides a suitable example of a Fischer matrices calculation. We use information from the character tables of the groups $2{\cdot}\mathrm{Fi}_{22}{:}2$ and $2 \times 2{\cdot}\mathrm{Fi}_{22}{:}2$ to check our calculations.

We write our group $\bar{G} = 2^2{\cdot}\mathrm{Fi}_{22}{:}2 \cong (2 \times 2{\cdot}\mathrm{Fi}_{22}){:}2$ as

$$(\langle x \rangle \times \langle z \rangle{\cdot}\mathrm{Fi}_{22}){:}\langle \sigma \rangle$$

so that σ acts to swap x with xz and $\langle x, z, \sigma \rangle \cong D_8$. $\bar{G}$ has three orbits on elements of the normal 2^2 group and the stabilisers are:

1. $\bar{G}$ fixing the identity. The trivial character of 2^2 extends to $\bar{G}$.

2. $2 \times 2{\cdot}\mathrm{Fi}_{22}$ fixing x and xz. Neither of the other two characters of the 2^2 that represent z faithfully can extend to the inertia group for the following reason: Choose $g \in 2^2{\cdot}\mathrm{Fi}_{22}$ that is conjugate to zg (which is possible since the extension is non-split) and let χ be such an extension. Then $\chi(g) = \chi(zg)$ which is non-zero as χ is linear. But z is represented as -1 which forces $\chi(g) = -\chi(zg)$.

3. $\bar{G}$ fixing z. The character that takes values -1 on x and xz extends to the inertia group.

The required inertia factors are therefore $\mathrm{Fi}_{22}{:}2$, Fi_{22}, and $\mathrm{Fi}_{22}{:}2$ respectively, and we must use the projective character table of Fi_{22}.

The program to construct the character table is attached (`ct22g2.gap`) which defines a function, `CharacterTableTwoSquaredGsplitTwo`, for constructing the character table of such a group.

The calculations are described in the following sections.

3 Conjugacy Classes and Fischer Matrices

We can easily find the number of conjugacy classes of $2^2{\cdot}\mathrm{Fi}_{22}{:}2$ lying above a class $[g]$ of $\mathrm{Fi}_{22}{:}2$ by counting class fusions from our inertia groups and using the fact that Fischer matrices are square. With a little more work we can also compute the class fusion from $2 \times 2{\cdot}\mathrm{Fi}_{22}$ which we use later to check our calculations.

We consider the diagram in Figure 1(a), and for $g \in 2{\cdot}\mathrm{Fi}_{22}{:}2$ write $\hat{g}$ for the image of g under the natural homomorphism to $\mathrm{Fi}_{22}{:}2$. The following cases arise:

1. Let $[\hat{g}]$ be a conjugacy class of Fi_{22} that does not fuse with another class under the action of σ.

$$\begin{tikzcd} \mathrm{Fi}_{22} \arrow[r, "\text{class fusion}"] \arrow[d, "\text{lift}"'] & \mathrm{Fi}_{22}{:}2 \arrow[d, "\text{lift}"] \\ 2{\cdot}\mathrm{Fi}_{22} \arrow[r, "\text{class fusion}"] & 2{\cdot}\mathrm{Fi}_{22}{:}2 \end{tikzcd}$$

(a) The groups.

$$\begin{tikzcd} 1A \arrow[r] \arrow[d] & 1A \arrow[d] \\ \begin{matrix} +1A \\ -1A \end{matrix} \arrow[r] & \begin{matrix} +1A \\ -1A \end{matrix} \end{tikzcd}$$

(b) As in case 1(a)i.

$$\begin{tikzcd} 2A \arrow[r] \arrow[d] & 2A \arrow[d] \\ \begin{matrix} +2A \\ -2A \end{matrix} \arrow[r] & \pm 2A \end{tikzcd}$$

(c) As in case 1(a)ii.

$$\begin{tikzcd} 16A \quad 16B \arrow[r] \arrow[d] & 16AB \arrow[d] \\ \pm 16A \quad \pm 16B \arrow[r] & \pm 16AB \end{tikzcd}$$

(d) As in case 2(a).

$$\begin{tikzcd} 11A \quad 11B \arrow[r] \arrow[d] & 11AB \arrow[d] \\ \begin{matrix} +11A & +11B \\ -11A & -11B \end{matrix} \arrow[r] & 11AB \quad -11AB \end{tikzcd}$$

(e) As in case 2(b).

$$\begin{tikzcd} 18A \quad 18B \arrow[r] \arrow[d] & 18AB \arrow[d] \\ \begin{matrix} +18A & +18B \\ -18A & -18B \end{matrix} \arrow[r] & \begin{matrix} +18A \sim -18B \\ -18A \sim +18B \end{matrix} \end{tikzcd}$$

(f) As in case 2(c).

Figure 1: The relationship between conjugacy classes of Fi_{22} and $2{\cdot}\mathrm{Fi}_{22}{:}2$. Horizontal arrows are for class fusion under the automorphism, and vertical arrows show lifting to the double cover. Diagram (a) shows the groups involved. The other diagrams give examples of some of the possibilities. Class names are for Fi_{22}.

(a) If g is not conjugate to zg in $2{\cdot}\mathrm{Fi}_{22}$ then:

i. If g is not conjugate to zg in $2{\cdot}\mathrm{Fi}_{22}{:}2$ (*e.g.*, $\hat{g} \in 1A$ as in Figure 1(b)) then we obtain three conjugacy classes with representatives g, xg, and zg respectively. (The second column corresponds to a class of twice the size of the other two, so xg and xzg must fuse to this class.) Our Fischer matrix is

$$M([\hat{g}]) = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 0 & -2 \\ 1 & -1 & 1 \end{pmatrix}$$

with $k = 4$ and $f_1 = 1, f_2 = 2, f_3 = 1$. The class $[\hat{g}]$ lifts to the first of these new classes and these elements have order $|g|$. If g has odd order then the other two classes contain elements of order $2|g|$, otherwise they contain elements of order $|g|$. (It is possible that $|g| = 2|\hat{g}|$. This case does not occur for Fi_{22}, the smallest ATLAS group where it does occur is $\mathrm{L}_2(17)$.)

ii. If g is conjugate to zg in $2{\cdot}\mathrm{Fi}_{22}{:}2$ (*e.g.*, $\hat{g} \in 2A$, as in Figure 1(c)) then we obtain three conjugacy class representatives xg, g, and xzg. Our Fischer matrix is

$$M([\hat{g}]) = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 0 & -2 \\ -1 & 1 & -1 \end{pmatrix}$$

with $k = 2$ and $f_1 = 1, f_2 = 2, f_3 = 1$. The class $[\hat{g}]$ must lift to the second of these new classes which is twice the size of the other two. The orders of the elements in both classes is the same as the order of g.

(b) i. Suppose that g is conjugate to zg in $2{\cdot}\mathrm{Fi}_{22}$ and g has the same order as $\hat{g}$ (*e.g.*, $\hat{g} \in 2C$). Then g and zg are also conjugate in $2{\cdot}\mathrm{Fi}_{22}{:}2$ and xg is conjugate to xzg in $\bar{G}$. We thus obtain 2 conjugacy classes and our Fischer matrix is

$$M([\hat{g}]) = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

with $k = 2$ and $f_1 = 1, f_2 = 1$. The class $[\hat{g}]$ lifts to the first of these new classes. Both new classes contain elements with the same order as $\hat{g}$.

ii. Suppose that g is conjugate to zg in $2{\cdot}\mathrm{Fi}_{22}$ and g has twice the order of $\hat{g}$. This case does not occur for Fi_{22}. An example is class $2A$ of A_5. The only difference from case 1(b)i is the element orders which are doubled.

2. Let $[\hat{g}]$ be a conjugacy class of Fi_{22} that is fused with a class $[\hat{h}]$ in $\mathrm{Fi}_{22}{:}2$.

(a) If g is conjugate to zg in $2{\cdot}\mathrm{Fi}_{22}$ (*e.g.*, $g \in 16A$ as in Figure 1(d)) then we obtain two conjugacy classes with representatives $g \sim zg \sim h \sim zh$ and $zg \sim xzg \sim xh \sim xzh$. The Fischer matrix is

$$M([\hat{g}]) = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

with $k = 2$ and $f_1 = 1, f_2 = 1$. and the class $[\hat{g}]$ lifts to the first of these new classes.

(b) Suppose that g is not conjugate to zg in $2{\cdot}\mathrm{Fi}_{22}$ and $[g]$ fuses with $[h]$ in $2{\cdot}\mathrm{Fi}_{22}{:}2$ (*e.g.*, $\hat{g} \in 11A$ and $\hat{h} \in 11B$, as in Figure 1(e)). It follows that $g \sim h$, $xg \sim xh$, $xzg \sim xzh$, and $zg \sim zh$ and these are representatives for the 4 new conjugacy classes *in that order*. Our Fischer matrix is

$$M([\hat{g}]) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$$

with $k = 4$ and $f_1 = f_2 = f_3 = f_4 = 1$. and 47.

(c) Suppose that g is not conjugate to zg in $2{\cdot}\mathrm{Fi}_{22}$ and $[g]$ fuses with $[zh]$ in $2{\cdot}\mathrm{Fi}_{22}{:}2$ (*e.g.*, $\hat{g} \in 18A$ and $\hat{h} \in 18B$, as in Figure 1(f)). The 4 new conjugacy classes are those of $xg \sim xzh$, $g \sim zh$, $zg \sim h$, and $xzg \sim xh$ in that order. The Fischer matrix is

$$M([\hat{g}]) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ -1 & 1 & 1 & -1 \end{pmatrix}$$

with $k = 4$ and $f_1 = f_2 = f_3 = f_4 = 1$.

3. Suppose now that $\hat{g} \in \mathrm{Fi}_{22}{:}2 \setminus \mathrm{Fi}_{22}$. Then $g^x = xgx = xxzg = zg$ so g is always conjugate to zg in G.

(a) Suppose that $\hat{g}$ has order indivisible by 4 and lifts to one class in $2{\cdot}\mathrm{Fi}_{22}{:}2$ of elements of the same order, *e.g.*, $\hat{g} \in 2D$. Now, $(xg)(xg) = zg^2$ which must have even order, so $\hat{g}$ lifts to 2 classes in $\bar{G}$, the second (with representatives xg and xzg) consisting of elements of twice the order of those in the first which have the same order as $\hat{g} \in \mathrm{Fi}_{22}{:}2$.

(b) If $\hat{g}$ has order indivisible by 4 and lifts to one class in $2{\cdot}\mathrm{Fi}_{22}{:}2$ of elements of twice the order, *e.g.*, $\hat{g} \in 2F$. This means that $(g)(g) = zg^2$ so $\hat{g}$ lifts to 2 classes in $\bar{G}$, the first (with representatives g and zg) consisting of elements of twice the order of those in the second which have the same order as $\hat{g} \in \mathrm{Fi}_{22}{:}2$.

(c) If $\hat{g}$ has order indivisible by 4 and lifts to two classes in $2{\cdot}\mathrm{Fi}_{22}{:}2$ of elements of the same order, *e.g.*, $\hat{g} \in 6M$, then $(zg)(zg) = g^2$ and $(xg)(xg) = zg^2$. Therefore $\hat{g}$ lifts to 2 classes in $\bar{G}$, the second (with representatives xg and xzg) consisting of elements of twice the order of those in the first which have the same order as $\hat{g} \in \mathrm{Fi}_{22}{:}2$.

(d) Suppose that $\hat{g}$ has order divisible by 4 and lifts to 2 classes of elements of the same order as $\hat{g}$, *e.g.*, $\hat{g} \in 8F$. Then $\hat{g}$ lifts to two conjugacy classes in $2{\cdot}\mathrm{Fi}_{22}{:}2$, both consisting of elements of the same order as $\hat{g} \in \mathrm{Fi}_{22}{:}2$.

(e) Suppose that $\hat{g}$ has order divisible by 4 and lifts to 2 classes of elements of the same order as $\hat{g}$. This case does not occur in Fi_{22}, an example is class $4A$ of A_5. In this case $\hat{g}$ lifts to two conjugacy classes in $2{\cdot}\mathrm{Fi}_{22}{:}2$, both consisting of elements of twice the order as $\hat{g} \in \mathrm{Fi}_{22}{:}2$.

In all of these cases the Fischer matrix is $\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ with $k = 2$ and $f_1 = f_2 = 1$.

4 From Class Functions To Characters

Our Fischer matrices are only defined up to multiplication of rows by -1, and in the 4×4 matrices any permutation of the bottom 3 rows is possible. However, we have chosen class fusion from $2 \times 2{\cdot}\mathrm{Fi}_{22}$ which forces us to use the matrices given above.

Using the Fischer matrices from above, and from GAP the character table of $\mathrm{Fi}_{22}{:}2$ and the projective character table of Fi_{22}, we assemble a table of class functions χ_i for $\bar{G}$ that obey row and column orthogonality. The element orders follow from the calculations above, as does class fusion from $2 \times 2{\cdot}\mathrm{Fi}_{22}$. Because of our choice of ordering of the conjugacy classes we can also write down the projection map to $2{\cdot}\mathrm{Fi}_{22}{:}2$

We use these maps to restrict each of our class functions to a class function ψ of the group $\mathrm{Fi}_{22}{:}2$. We then check that $\langle \psi, \chi \rangle \in \mathbb{N} \cup \{0\}$ for all $\chi \in \mathrm{Irr}(\mathrm{Fi}_{22}{:}2)$. (Whenever it was not, the reason was always because we had not used the correct Fischer matrix for our chosen ordering of the conjugacy classes.)

5 Power Maps

To compute the power maps we observe that classes lying above $[\hat{g}]$ must power up to classes lying above $[\hat{g}^p]$ for all p, and for odd p elements in $[ng]$ must p-power to elements in $[ng^p]$ for all $n \in \langle x, z\rangle$. For $p = 2$ and $\hat{g} \in \mathrm{Fi}_{22}$ elements in $[ng]$ square to elements in $[g]$ and for outer elements the 2-power map is clear from item 3 of the case analysis.

We can also use GAP to compute possible power maps from the character table. Unusually, it produces unique p-power maps for our table and these agree with ours. Furthermore, these agree with the power maps of $2 \times 2{}^{\cdot}\mathrm{Fi}_{22}$.

To further test our class functions and power maps we check that all symmetric and anti-symmetric parts of all irreducibles have non-negative inner products with all irreducibles.

Finally, GAP produces four possible class fusions to Fi_{24}. There are two independent choices.

- There are two classes of elements of order 26 lying above class $13A$ of $\mathrm{Fi}_{22}{:}2$. These could fuse either way round to the algebraically conjugate classes $26B$ and $26C$ of Fi_{24}.
- Our labelling of the involutions x and xz was arbitrary, if we swap them then our choice of class representatives in cases 1(a)ii, 2(b), 2(c) is effected. For example in case 2(b) the class representatives would become (in order) $g \sim h$, $xzg \sim xzh$, $zg \sim zh$, and $zg \sim zh$. The labelling of x and xz corresponds to the other choice for class fusion. (The classes of elements of order 4 lying above $11A$ are not effected since they fuse to the same class in Fi_{24}.)

6 A GAP Function For The General Case

The function

```
CharacterTablleTwoSquaredGsplitTwo(t_g, t_g2, t_2g, t_2g2, proj1, proj2)
```

assembles the character table. The arguments are:

1. `t_g` The character table of G.
2. `t_g2` The character table of $G{:}2$.
3. `t_2g` The character table of $2{}^{\cdot}G$.
4. `t_2g2` The character table of $2{}^{\cdot}G{:}2$.
5. `proj1` The index in `ProjectivesInfo(t_g)` for the record with name $2{}^{\cdot}G$. This is usually 1.
6. `proj2` The index in `ProjectivesInfo(t_g2)` for the record with name $2{}^{\cdot}G{:}2$. This is usually 1.

7 Conclusions

Finally, a word of warning. We have always assumed that in $2{}^{\cdot}G{:}2$ the normal subgroup $2{}^{\cdot}G$ has a complement. However, this need not always be the case: A_6 has three involutory automorphisms, σ, τ, and $\rho = \sigma\tau$ say, with

$$\mathrm{A}_6{:}\langle\sigma\rangle \cong \mathrm{S}_6 \qquad \mathrm{A}_6{:}\langle\tau\rangle \cong \mathrm{PGL}_2(9) \qquad \mathrm{A}_6{}^{\cdot}\langle\rho\rangle \cong \mathrm{M}_{10}$$

Our program can produce character tables for $2^2{}^\cdot S_6$ and $2^2{}^\cdot PGL_2(9)$, but not for $2^2{}^\cdot M_{10}$. This is because there is no group $2{}^\cdot A_6.2_3$, just a group $(4 \circ 2{}^\cdot A_6){}^\cdot 2_3$ which is isoclinic to $(2 \times 2{}^\cdot A_6){}^\cdot 2_3$.

References

[AM03] Faryad Ali and Jamshid Moori, *The Fischer-Clifford matrices of a maximal subgroup of* Fi'_{24}, Represent. Theory **7** (2003), 300–321 (electronic). MR MR1993362 (2004i:20023)

[CCN+85] J. H. Conway, R. T. Curtis, S. P. Norton, R. A. Parker, and R. A. Wilson, *An ATLAS of finite groups*, Oxford University Press, Eynsham, 1985, Maximal subgroups and ordinary characters for simple groups, With computational assistance from J. G. Thackray.

[Fis91] B. Fischer, *Clifford matrices*, Representation Theory of Finite Groups and Finite-Dimensional Lie Algebras, 1991, pp. 1–16.

[GAP02] The GAP Group, *GAP – Groups, Algorithms, and Programming, Version 4.3*, 2002, (`www.gap-system.org`).

[Isa94] I. M. Isaacs, *Character theory of finite groups*, Dover, 1994.

[Kar85] G. Karpilovsky, *Projective representations of finite groups*, Monographs and textbooks in pure and applied mathematics, no. 94, Marcel Dekker, 1985.