Chapter 14

MSM2G2 Advanced Calculus

(14.1) Second Order Linear Differential Equations

(14.1.1) Constant Coefficients

Differential equations of the form

$$\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)y = f(x)$$

appear commonly in applied mathematics. The simplest case is when the coefficients are constants, and it is clear that when $f(x) = 0$, the solution is of the form $y = e^{mx}$. Substituting this in gives $(m^2 + 2a_1 m + a_0)e^{mx} = 0$ which has two solutions for m. The solution to the differential equation is therefore $y = Ae^{m_1 x} + Be^{m_2 x}$. Note however, that

1. when m_1 and m_2 are real distinct roots, the solution is of the form $y = Ae^{m_1 x} + Be^{m_2 x}$.
2. when the root is repeated, i.e. $m_1 = m_2$ then the solution is of the form $y = (A + Bx)e^{m_1 x}$.
3. when the solutions are complex i.e. $m_1 = p + iq$ and $m_2 = p - iq$ then the solution can be expressed in the form $y = e^{px}(A\cos qx + B\sin qx)$.

The two solutions to the homogeneous equation form part of the solution to any non-homogeneous variant of the same differential equation. A particular integral is usually found by 'guessing' at a function with unknown coefficients, then substituting it in to find the coefficients.

It is found that if $a_1(x)$, $a_0(x)$, and $f(x)$ are all continuous on the open interval I and $a \in I$, then a unique solution to the initial value problem exists on I. By an initial value problem, it is meant that the boundary conditions $y(a) = \alpha$ and $y'(a) = \beta$ are given.

(14.1.2) Coefficients As A Function Of x, The Independent Variable

When the coefficients are not constant there is a lot more work to do in order to find a solution. There is one special case where the differential equation of of the form

$$x^2\frac{d^2y}{dx^2} + a_1 x\frac{dy}{dx} + a_0 y = 0$$

These are known as Euler-Cauchy equations, and the solutions are of the form $y = x^\alpha$, with α being found by substituting into the equation and solving the resulting quadratic for α. Alternatively, a transformation such as $z = \ln|x|$ can be used. By the chain rule,

$$\frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx} = \frac{dy}{dz}\frac{1}{x}$$

and

$$\begin{aligned}\frac{\mathrm{d}^2y}{\mathrm{d}x^2} &= \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}y}{\mathrm{d}z}\frac{1}{x}\right)\\ &= \frac{\mathrm{d}^2y}{\mathrm{d}z^2}\frac{\mathrm{d}z}{\mathrm{d}x}\frac{1}{x} - \frac{\mathrm{d}^2y}{\mathrm{d}z^2}\frac{1}{x^2}\end{aligned}$$

Substituting these back into the differential equation produces an equation with constant coefficients.

(14.1.3) Reduction Of Order

The method of reduction of order works on the assumption that one solution is already known. This is not as unrealistic as it sounds, as it is sometimes easy to find one of the particular solutions by inspection. Consider the differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + a_1\frac{\mathrm{d}y}{\mathrm{d}x} + a_0(x)y = 0 \tag{1}$$

and suppose that one of the solutions, $u(x)$ say, is known. Now suppose that the other solution is of the form $y(x) = U(x)u(x)$, then

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= u\frac{\mathrm{d}U}{\mathrm{d}x} + U\frac{\mathrm{d}u}{\mathrm{d}x}\\ \frac{\mathrm{d}^2y}{\mathrm{d}x^2} &= u\frac{\mathrm{d}^2U}{\mathrm{d}x^2} + 2\frac{\mathrm{d}U}{\mathrm{d}x}\frac{\mathrm{d}u}{\mathrm{d}x} + U\frac{\mathrm{d}^2u}{\mathrm{d}x^2}\end{aligned}$$

Now substituting back into (1),

$$\begin{aligned}u\frac{\mathrm{d}^2U}{\mathrm{d}x^2} + 2\frac{\mathrm{d}U}{\mathrm{d}x}\frac{\mathrm{d}u}{\mathrm{d}x} + U\frac{\mathrm{d}^2u}{\mathrm{d}x^2} + a_1\left(u\frac{\mathrm{d}U}{\mathrm{d}x} + U\frac{\mathrm{d}u}{\mathrm{d}x}\right) + a_0Uu = 0\\ U\left(\frac{\mathrm{d}^2u}{\mathrm{d}x^2} + a_1\frac{\mathrm{d}u}{\mathrm{d}x} + a_0\right) + \frac{\mathrm{d}U}{\mathrm{d}x}\left(2\frac{\mathrm{d}u}{\mathrm{d}x} + a_1u\right) + \frac{\mathrm{d}^2U}{\mathrm{d}x^2}u = 0\end{aligned}$$

But u is a solution to (1), so using this and putting $z = \frac{\mathrm{d}U}{\mathrm{d}x}$,

$$\begin{aligned}u\frac{\mathrm{d}z}{\mathrm{d}x} + z\left(2\frac{\mathrm{d}u}{\mathrm{d}x} + a_1u\right) &= 0\\ \frac{1}{z}\frac{\mathrm{d}z}{\mathrm{d}x} + \frac{2}{u}\frac{\mathrm{d}u}{\mathrm{d}x} + a_1 &= 0\end{aligned}$$

now integrating

$$\begin{aligned}\ln|z| + 2\ln|u| + \int a_1(x)\,dx = c\\ \frac{\mathrm{d}U}{\mathrm{d}x} = z = \frac{c}{u^2}\exp\left(-\int^x a_1(s)\,ds\right)\\ U(x) = \int^x \frac{c}{(u(t))^2}\exp\left(-\int^t a_1(s)\,ds\right)\,dt + c_1\end{aligned}$$

giving the second solution to be

$$y(x) = U(x)u(x) = u(x)\int^x \frac{c}{(u(t))^2}\exp\left(-\int^t a_1(s)\,ds\right)\,dt + c_1$$

This is called the reduction of order formula, although it is preferable to learn the method rather than just the formula.

(14.1.4) Variation Of Parameters

The reduction of order method makes it possible to find one part of the homogeneous solution to a differential equation given the other part. In the general case, what now remains is to find the particular solution—and hence the complete solution to the differential equation— and this is the purpose of the variation of parameters method. The variation of parameters works by 'toying' with the constant multiples of the two parts of the homogeneous solution. In seeking solutions of the differential equation

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + a_1 \frac{\mathrm{d}y}{\mathrm{d}x} + a_0 y = f(x) \tag{2}$$

which has known homogeneous solution as a linear combination of $u(x)$ and $v(x)$. Suppose that the coefficients of these functions are variable, and so put

$$y = c(x)u(x) + d(x)v(x)$$

and so

$$\frac{\mathrm{d}y}{\mathrm{d}x} = c\frac{\mathrm{d}u}{\mathrm{d}x} + u\frac{\mathrm{d}c}{\mathrm{d}x} + d\frac{\mathrm{d}v}{\mathrm{d}x} + v\frac{\mathrm{d}d}{\mathrm{d}x}$$

Since in fact c and d are (supposed to be) constant, they can be chosen at will, and hence choose them so that

$$u\frac{\mathrm{d}c}{\mathrm{d}x} + v\frac{\mathrm{d}d}{\mathrm{d}x} = 0 \tag{3}$$

indeed, if c and d are constant, then this is the case anyway. This gives

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} &= c\frac{\mathrm{d}u}{\mathrm{d}x} + d\frac{\mathrm{d}v}{\mathrm{d}x} \quad \text{and} \\
\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} &= c\frac{\mathrm{d}^2 u}{\mathrm{d}x^2} + \frac{\mathrm{d}c}{\mathrm{d}x}\frac{\mathrm{d}u}{\mathrm{d}x} + d\frac{\mathrm{d}^2 v}{\mathrm{d}x^2} + \frac{\mathrm{d}d}{\mathrm{d}x}\frac{\mathrm{d}v}{\mathrm{d}x}
\end{aligned}$$

Now substituting into (2),

$$\begin{aligned}
c\frac{\mathrm{d}^2 u}{\mathrm{d}x^2} + \frac{\mathrm{d}c}{\mathrm{d}x}\frac{\mathrm{d}u}{\mathrm{d}x} + d\frac{\mathrm{d}^2 v}{\mathrm{d}x^2} + \frac{\mathrm{d}d}{\mathrm{d}x}\frac{\mathrm{d}v}{\mathrm{d}x} + a_1\left(c\frac{\mathrm{d}u}{\mathrm{d}x} + d\frac{\mathrm{d}v}{\mathrm{d}x}\right) + a_0(cu + dv) &= f \\
c\left(\frac{\mathrm{d}^2 u}{\mathrm{d}x^2} + a_1\frac{\mathrm{d}u}{\mathrm{d}x} + a_0 u\right) + d\left(\frac{\mathrm{d}^2 v}{\mathrm{d}x^2} + a_1\frac{\mathrm{d}v}{\mathrm{d}x} + a_0 v\right) + \frac{\mathrm{d}u}{\mathrm{d}x}\frac{\mathrm{d}c}{\mathrm{d}x} + \frac{\mathrm{d}v}{\mathrm{d}x}\frac{\mathrm{d}d}{\mathrm{d}x} &= f
\end{aligned}$$

but the contents of the first two brackets are of the form of (2), for which u and v are solutions to the homogeneous form, hence

$$\frac{\mathrm{d}u}{\mathrm{d}x}\frac{\mathrm{d}c}{\mathrm{d}x} + \frac{\mathrm{d}v}{\mathrm{d}x}\frac{\mathrm{d}d}{\mathrm{d}x} = f \tag{4}$$

Equations (3) and (4) are two linear equations in $\frac{\mathrm{d}c}{\mathrm{d}x}$ and $\frac{\mathrm{d}d}{\mathrm{d}x}$ which can be expressed in the usual way as

$$\begin{pmatrix} u & v \\ u' & v' \end{pmatrix}\begin{pmatrix} c' \\ d' \end{pmatrix} = \begin{pmatrix} 0 \\ f \end{pmatrix}$$

Where $\mathcal{W} = \begin{vmatrix} u & v \\ u' & v' \end{vmatrix}$, the solutions to these equations are

$$c' = -\frac{fv}{\mathcal{W}} \qquad \text{and} \qquad d' = \frac{fu}{\mathcal{W}} \tag{5}$$

The determinant $\mathcal{W}$ is called the Wronskian, and its uses and properties are discussed later. Note that here it is required that $\mathcal{W} \neq 0$. Integrating (5) gives the solutions,

$$\begin{aligned} y(x) &= u(x) \int^x -\frac{f(s)v(s)}{\mathcal{W}}\, ds + v(x) \int^x \frac{f(s)u(s)}{\mathcal{W}}\, ds \\ &= \int^x f(s) \frac{u(s)v(x) - u(x)v(s)}{\mathcal{W}}\, ds \end{aligned} \tag{6}$$

This is called the variation of parameters formula. It would be possible to memorise the formula, but it is preferable to use the method.

(14.1.5) Uses Of The Wronskian

In a similar way to vectors being linearly independent as described in Chapter **??**, functions can the thought of as being linearly independent.

Definition 7 *Let u and v be C_1 functions and $a, b \in \mathbb{R}$. u and v are linearly independent if the only solution to $au(x) + bv(x) = 0$ is $a = b = 0$ for all x for which u and v are defined.*

If the relationship $au(x) + bv(x) = 0$ is differentiated, then it gives $au'(x) + bv'(x) = 0$, and the two equations can be written in matrix form as

$$\begin{pmatrix} u(x) & v(x) \\ u'(x) & v'(x) \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{8}$$

There are only non-trivial solutions when the determinant of the Wronskian is zero. Note that the Wronskian is usually a function of x, and this is allowed to take value zero — the point is that the Wronskian is not identically zero.

(14.1.6) Frobenius' Method

If one of the homogeneous solutions is not already known, then the methods described above are not much use. Frobenius method provides a way of finding the homogeneous solutions as series. The solution is assumed to be of the form

$$y = x^c \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{n+c} \tag{9}$$

There is no hard and fast way to give a method, so Frobenius' Method is introduced by means of an example.

Example 10 *Bessel's equation with $v = \frac{1}{2}$ is*

$$x^2 \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + x \frac{\mathrm{d}y}{\mathrm{d}x} + \left(x^2 - \frac{1}{4}\right) y = 0 \tag{11}$$

Proof. Solution Using the Frobenius solution series,

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^{n+c} \\ \frac{\mathrm{d}y}{\mathrm{d}x} &= \sum_{n=1}^{\infty} a_n (n+c) x^{n+c-1} \\ \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} &= \sum_{n=0}^{\infty} a_n (n+c)(n+c-1) x^{n+c-2} \end{aligned}$$

These are now substitutes back into the differential equation (11).

$$x^2 \sum_{n=0}^{\infty} a_n(n+c)(n+c-1)x^{n+c-2} + x \sum_{n=0}^{\infty} a_n(n+c)x^{n+c-1} + \left(x^2 - \frac{1}{4}\right) \sum_{n=2}^{\infty} a_n x^{n+c} = 0$$

now collect the summations

$$\begin{aligned}\sum_{n=0}^{\infty} a_n \left((n+c)(n+x-1) + (n+c) - \frac{1}{4}\right) x^{n+c} + \sum_{n=0}^{\infty} a_n x^{n+c+2} &= 0 \\ \sum_{n=0}^{\infty} a_n \left((n+c)^2 - \frac{1}{4}\right) x^{n+c} + \sum_{n=0}^{\infty} a_n x^{n+c+2} &= 0\end{aligned}$$

These sums are clearly very similar, and it is preferable to combine them. In order to do this, the first step is to extract the first two terms of the first sum, giving

$$a_0 \left(c^2 - \frac{1}{4}\right) x^c + a_1 \left((c+1)^2 - \frac{1}{4}\right) x^{c+1} + \sum_{n=2}^{\infty} a_n \left((n+c)^2 - \frac{1}{4}\right) x^{n+c} + \sum_{n=0}^{\infty} a_n x^{n+c+2} = 0$$

In the second sum, put $m = n+2$ so $n = m-2$ to give

$$a_0 \left(c^2 - \frac{1}{4}\right) x^c + a_1 \left((c+1)^2 - \frac{1}{4}\right) x^{c+1} + \sum_{n=2}^{\infty} a_n \left((n+c)^2 - \frac{1}{4}\right) x^{n+c} + \sum_{m=2}^{\infty} a_{m-2} x^{m+c} = 0$$

The variables over which the sum is taken are arbitrary, so this can be re-written as

$$a_0 \left(c^2 - \frac{1}{4}\right) x^c + a_1 \left((c+1)^2 - \frac{1}{4}\right) x^{c+1} + \sum_{n=2}^{\infty} a_n \left((n+c)^2 - \frac{1}{4}\right) x^{n+c} + \sum_{n=2}^{\infty} a_{n-2} x^{n+c} = 0$$

and hence

$$a_0 \left(c^2 - \frac{1}{4}\right) x^c + a_1 \left((c+1)^2 - \frac{1}{4}\right) x^{c+1} + \sum_{n=2}^{\infty} a_n \left(\left((n+c)^2 - \frac{1}{4}\right) + a_{n-2}\right) x^{n+c} = 0 \tag{12}$$

Now, in order for all this to be equal to zero, each coefficient of each power of x must be zero. To this point, all Frobenius analysis follows the same form: it is now that differences appear. Taking the lowest power of x first,

$$\begin{aligned}a_0 \left(c^2 - \frac{1}{4}\right) = 0 \quad \text{so} \quad c^2 - \frac{1}{2} &= 0 \\ c &= \pm\frac{1}{2}\end{aligned} \tag{13}$$

This solution means that a_1 could or could not be zero. a_0 is never zero, as if it was then the term with x^{c+1} would be the first term, and c would be 'redefined' to reflect this — if ever $a_0 = 0$ then all the terms 'shuffle down' so that in fact it isn't. It is also required that $a_1 \neq 0$, so it must be the case that $c = \frac{-1}{2}$. This has taken care of the first two powers of x. The others are now dealt with by substituting $c = \frac{-1}{2}$ into (12), giving

$$\begin{aligned}a_n \left(\left(n - \frac{1}{2}\right)^2 - \frac{1}{4}\right) + a_{n-2} &= 0 \\ a_n \left(n^2 - n\right) &= -a_{n-2} \\ a_n &= \frac{-a_{n-2}}{n(n-1)}\end{aligned} \tag{14}$$

This recurrence relation must now be solved in order to find the series solution. This can usually be done 'by inspection' — i.e. by spotting the pattern, or guessing as pure mathematicians call it — and in this case, the following happens.

$$\begin{aligned} a_2 &= \frac{-a_0}{2!} \\ a_3 &= \frac{-a_1}{3!} \\ a_4 &= \frac{-a_2}{4.3} = \frac{a_0}{4!} \\ a_5 &= \frac{-a_3}{5.4} = \frac{a_1}{5!} \end{aligned}$$

it is evident that

$$\begin{aligned} a_{2n} &= \frac{(-1)^n a_0}{(2n)!} \\ a_{2n+1} &= \frac{(-1)^n a_1}{(2n+1)!} \end{aligned}$$

Now substituting all this back into the original equation (9), for the power series,

$$\begin{aligned} y &= x^c \sum_{n=0}^{\infty} a_n x^n \\ &= x^{\frac{-1}{2}} \left(a_0 + a_1 x + a_2 x^2 + \dots \right) \\ &= x^{\frac{-1}{2}} a_0 \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \dots \right) + x^{\frac{-1}{2}} a_1 \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} + \dots \right) \\ &= a_0 \frac{\cos x}{\sqrt{x}} + a_1 \frac{\sin x}{\sqrt{x}} \end{aligned}$$

The two parts of this solution are linearly independent, and are important equations which will be discussed later. □

This example shows only one way in which calculations can proceed. Equation (13) is called the Indicial equation, and it is key in determining the Frobenius' method solution to the equation.

Theorem 15 (Frobenius General Rule 1) *Suppose the indicial equation has two roots, $c = \alpha$ and $c = \beta$ where $\alpha < \beta$ and the difference between them is an integer. If one of the coefficients becomes indeterminate by putting $c = \alpha$, then both solutions can be generated by putting $c = \alpha$ to find a recurrence relation.*

The next situation is now introduced by means of another example.

Example 16 *Use an infinite power series to find the solutions to*

$$2x(1-x)\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + (1-x)\frac{\mathrm{d}y}{\mathrm{d}x} + 3y = 0 \qquad (17)$$

Proof. Solution Using the Frobenius power series,

$$y = \sum_{n=0}^{\infty} a_n x^{n+c}$$
$$\frac{\mathrm{d}y}{\mathrm{d}x} = \sum_{n=1}^{\infty} a_n (n+c) x^{n+c-1}$$
$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \sum_{n=0}^{\infty} a_n (n+c)(n+c-1) x^{n+c-2}$$

These are now substituted back into the differential equation (17).

$$2x(1-x)\left(\sum_{n=0}^{\infty} a_n (n+c)(n+c-1)x^{n+c-2}\right) + (1-x)\left(\sum_{n=1}^{\infty} a_n (n+c)x^{n+c-1}\right) + 3\sum_{n=1}^{\infty} a_n (n+c)x^{n+c-1} = 0$$

Now collect terms with the same power of x,

$$\sum_{n=0}^{\infty} a_n \left(2(n+x)(n+c-1) + (n+c)\right) x^{n+c-1} + \sum_{n=0}^{\infty} a_n \left(3 - 2(n+c)(n+c-1) - (n+c)\right) x^{n+c} = 0$$
$$\sum_{n=0}^{\infty} a_n (n+c)(2n+2n-1)x^{n+c-1} + \sum_{n=0}^{\infty} a_n \left(3 - (n+c)(2n+2c-1)\right) x^{n+c} = 0$$
$$a_0 c(2c-1)x^{c-1} + \sum_{n=1}^{\infty} a_n (n+c)(2n+2n-1)x^{n+c-1} + \sum_{n=0}^{\infty} a_n \left(3 - (n+c)(2n+2c-1)\right) x^{n+c} = 0$$

In the second summation, put $m = n+1$, then since the variables of summation are irrelevant, combine the sums to give

$$a_0 c(2c-1)x^{c-1} + \sum_{n=1}^{\infty} \left(a_n (n+c)(2n+2c-1) + a_{n-1}\left(3 - (n-1+c)(2n+2c-3)\right)\right) x^{n+c-1} = 0$$

The coefficients of the powers of x are set to zero, so the indicial equation is

$$a_0 c(2c-1) = 0$$

It is required that $a_0 \neq 0$ and so the solutions are

$$c = 0 \qquad c = \frac{1}{2}$$

Hence by considering the general summand,

$$a_n = a_{n-1}\frac{(n-1+c)(2n+2c-3) - 3}{(n+c)(2n+2c-1)}$$

For $c = 0$,

$$\begin{aligned} a_n &= a_{n-1}\frac{(n-1)(2n-3)-3}{n(2n-1)} \\ \text{so} \quad a_1 &= \frac{-3}{1}a_0 \\ a_2 &= \frac{-1}{3}a_1 = \frac{3a_0}{3} \\ a_3 &= \frac{1}{5}a_2 = \frac{3a_0}{5.3} \\ a_4 &= \frac{3}{7}a_3 = \frac{3a_0}{7.5} \\ a_5 &= \frac{5}{9}a_3 = \frac{3a_0}{9.7} \end{aligned}$$

It is evident that

$$a_n = \frac{3a_0}{(2n-1)(2n-3)}$$

Hence one of the solutions is

$$y = 3a_0 \sum_{n=0}^{\infty} 3a_0 \frac{x^n}{(2n-1)(2n-3)}$$

It would now be possible to use reduction of order to find the other solution. However, since the difference between the solutions to the indicial equation is not an integer, the other value of c can be used to produce the other solution. This doesn't work when there is an integer difference because the two series would 'overlap'. Now taking $c = \frac{1}{2}$,

$$\begin{aligned} a_n &= a_{n-1}\frac{\left(n-\frac{1}{2}\right)(2n-2)-3}{\left(1+\frac{1}{2}\right)2n} \\ &= a_{n-1}\frac{2n^2-3n-2}{2n^2+1} \\ &= a_{n-1}\frac{n-2}{n} \\ a_1 &= -a_0 \\ a_2 &= \frac{0}{2}a_1 = 0 \\ a_3 &= \frac{1}{2}a_2 = 0 \end{aligned}$$

So $a_n = 0$ for $n \geqslant 2$. The second solution is therefore

$$\begin{aligned} y &= x^{\frac{1}{2}} \sum_{n=0}^{\infty} a_n x^n \\ &= a_0 x^{\frac{1}{2}}(1-x) \end{aligned}$$

□

This kind of solution is an example of the next part of Frobenius method.

Theorem 18 (Frobenius General Rule 2) *If the indicial equation has two roots, α and β, ($\alpha < \beta$) with non-integer difference, then the two solutions are found by substituting $c = \alpha$ then $c = \beta$ into the recurrence relation.*

The third case is when the indicial equation has a repeated root.

Example 19 *Given that the differential equation*

$$x\frac{d^2y}{dx^2} + (1+x)\frac{dy}{dx} + 2y = 0$$

becomes

$$a_0c^2x^{c-1} + \sum_{n=1}^{\infty}\left(a_n(n+c)^2 + a_{n-1}(n+c+1)\right)x^{n+c} = 0$$

when the Frobenius power series is substituted in, find the two independent solutions.

Proof. Solution The indicial equation is $c^2 = 0$, and the recurrence relation is

$$a_n = a_{n-1}\frac{-(n+c+1)}{(n+c)^2}$$

Using $c = 0$,

$$\begin{aligned} a_n &= a_{n-1}\frac{-(n+1)}{n^2} \\ \text{so} \quad a_1 &= \frac{-2}{1^2}a_0 \\ a_2 &= \frac{-3}{2^2}a_1 = \frac{3.2}{1^2 2^2}a_0 \\ a_3 &= \frac{-4}{3^2}a_1 = \frac{-4.3.2}{1^2 2^2 3^2}a_0 \end{aligned}$$

it is easy to see that

$$\begin{aligned} a_n &= \frac{(-1)^n(n+1)!}{(n!)^2} \\ &= \frac{(-1)^n(n+1)}{n!} \end{aligned}$$

So one of the solutions is

$$y = a_0\sum_{n=0}^{\infty}\frac{(-1)^n(n+1)x^n}{n!}$$

From substituting in the Frobenius series,

$$a_0c^2x^{c-1} + \sum_{n=1}^{\infty}\left(a_n(n+c)^2 + a_{n-1}(n+c-1)\right)x^{n+c} = 0 \quad (20)$$

and since the purpose of this method is to find the coefficients of the powers of x to be zero, the obvious next move is to choose

$$a_n(n+c)^2 + a_{n-1}(n+c+1) = 0$$

where a_n is a function of c. Equation (20) now becomes $a_0c^2x^{c-1} = 0$, and so

$$x\frac{d^2y}{dx^2} + (1+x)\frac{dy}{dx} + 2y = a_0c^2x^{c-1} \quad (21)$$

Hence y is now a function of both x and c. Equation (21) is now differentiated with respect to c. First note that

$$\frac{\partial}{\partial c}\left(\frac{\partial y}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial y}{\partial c}\right)$$

so

$$
\begin{aligned}
x\frac{\partial^2}{\partial x^2}\left(\frac{\partial y}{\partial c}\right) + (1+x)\frac{\partial}{\partial x}\left(\frac{\partial y}{\partial c}\right) + 2\frac{\partial y}{\partial c} &= a_0\frac{\partial}{\partial c}(c^2x^{c-1}) \\
&= c^2\frac{\partial}{\partial c}(x^{c-1}) + x^{c-1}\frac{\partial}{\partial c}(c^2) \\
&= c^2\frac{\partial}{\partial c}\left(\frac{1}{x}e^{c\ln x}\right) + x^{c-1}\frac{\partial}{\partial c}(c^2) \\
&= c^2e^{c\ln x}\frac{1}{x}\ln x + 2cx^{c-1} \\
&= c^2x^{c-1}\ln x + 2cx^{c-1}
\end{aligned}
$$

Now let $c \to 0$,

$$
x\frac{\partial^2}{\partial x^2}\left(\frac{\partial y}{\partial c}\right)_{c=0} + (1+x)\frac{\partial}{\partial x}\left(\frac{\partial y}{\partial c}\right)_{c=0} + 2\left(\frac{\partial y}{\partial c}\right)_{c=0} = 0
$$

From this equation, it is evident that $\left(\frac{\partial y}{\partial c}\right)_{c=0}$ is a solution to the differential equation. To find this solution, return to the Frobenius series,

$$
\begin{aligned}
y &= x^c\sum_{n=0}^{\infty} a_nx^n \quad \text{remember} \quad a_n(n+c)^2 + a_{n-1}(n+c+1) = 0 \\
\frac{\partial y}{\partial c} &= x^c\sum_{n=0}^{\infty}\frac{\mathrm{d}a_n}{\mathrm{d}c}x^n + \sum_{n=0}^{\infty} a_nx^nx^c\ln x \\
\left(\frac{\partial y}{\partial c}\right)_{c=0} &= \sum_{n=0}^{\infty} x^n\left(\frac{\mathrm{d}a_n}{\mathrm{d}c}\right)_{c=0} + \left(\sum_{n=0}^{\infty} x^n(a_n)_{c=0}\right)\ln x
\end{aligned}
$$

This is the solution. However, it is necessary to find $\left(\frac{\mathrm{d}a_n}{\mathrm{d}c}\right)_{c=0}$. Going back to the recurrence relation,

$$
\begin{aligned}
a_n(n+c)^2 + a_{n-1}(n+c+1) = 0& \\
a_n &= -\frac{n+c+1}{(n+c)^2} \\
a_1 &= -a_0\frac{c+2}{(1+c)^2} \\
a_2 &= -a_1\frac{c+3}{(2+c)^2} = a_0\frac{(c+2)(c+3)}{(1+c)^2(2+c)^2}
\end{aligned}
$$

from which it is evident that

$$
a_n = \frac{(-1)^na_0(c+n+1)!}{((c+n)!)^2}
$$

This now needs to be differentiated. However, to make this possible without excessive manipulation, logarithmic differentiation is used. This gives.

$$\begin{aligned}
\frac{1}{a_n}\frac{\mathrm{d}a_n}{\mathrm{d}c} &= \sum_{i=1}^{n}\frac{1}{c+i+1} - 2\sum_{i=1}^{n}\frac{1}{c+i} \\
\frac{\mathrm{d}a_n}{\mathrm{d}c} &= \frac{(-1)^n a_0 (c+n+1)!}{((c+n)!)^2}\left(\sum_{i=1}^{n}\frac{1}{c+i+1} - 2\sum_{i=1}^{n}\frac{1}{c+i}\right) \\
\left(\frac{\mathrm{d}a_n}{\mathrm{d}c}\right)_{c=0} &= (-1)^n\frac{(n+1)!}{(n!)^2}a_0\left(\sum_{i=1}^{n}\frac{1}{i+1} - 2\sum_{i=1}^{n}\frac{1}{i}\right)
\end{aligned}$$

The second solution to the differential equation is, therefore,

$$y = a_0\left(\sum_{n=0}^{\infty}(-1)^n\frac{n+1}{n!}\left(\phi(n+1) - 2\phi(n) - 1\right)x^n + \ln x\sum_{n=0}^{\infty}(-1)^n\frac{n+1}{n!}x^n\right)$$ □

This rather laborious method is little used, and is the final part of Frobenius method.

Theorem 22 (Frobenius General Rule 3) *If the indicial equation has a repeated root $c = \alpha$, then one solution is obtained by putting $c = \alpha$ in the recurrence relation. The second solution is $\left(\frac{\mathrm{d}y}{\mathrm{d}c}\right)_{c=\alpha}$, where a_n is is treat as a function of c.*

(14.1.7) Validity Of Frobenius Method

It is all very well finding series solutions, however, finding such a series says nothing about whether it converges.

Generally, a second order differential equation can be written in the form

$$P(x)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + Q(x)\frac{\mathrm{d}y}{\mathrm{d}x} + R(x)y = 0 \tag{23}$$

However, it is usual to divide through by $P(x)$ so clearly there is a problem if $P(x) = 0$.

Definition 24 *If $P(x_0) \neq 0$ then x_0 is an ordinary point of the differential equation. If however $P(x_0) = 0$ then x_0 is a singular point of the differential equation.*

Singular points are not a particular problem, as long as they are 'well behaved'. A singular point x_0 is called regular if

$$(x - x_0)\frac{Q(x)}{P(x)} \quad \text{and} \quad (x - x_0)^2\frac{R(x)}{P(x)}$$

both have convergent Taylor series expansions. If either of these conditions fail, then the singular point is called irregular.

Suppose x_0 is an ordinary point and the two power series

$$\frac{Q(x)}{P(x)} \quad \text{and} \quad \frac{R(x)}{P(x)}$$

are both convergent for $|x - x_0| < \rho$. In this case the Frobenius power series solution exists, is unique, and is convergent for $|x - x_0| < \rho$.

Suppose that x_0 is a regular singular point and that the power series

$$(x - x_0)\frac{Q(x)}{P(x)} \quad \text{and} \quad (x - x_0)^2\frac{R(x)}{P(x)}$$

are both convergent for $|x - x_0| < \rho$. Then the Frobenius power series solution exists, is unique, and is convergent for $|x - x_0| < \rho$.

At irregular singular points, it is not possible to determine whether the Frobenius series solution will work.

(14.2) Legendre Functions

(14.2.1) Legendre's Equation

Many physical events can be described by using Legendre's equation,

$$(1 - x^2)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 2x\frac{\mathrm{d}y}{\mathrm{d}x} + n(n+1)y = 0 \tag{25}$$

This can be solved using Frobenius' method as follows.

$$\begin{aligned}
(1 - x^2)\sum_{i=0}^{\infty} a_i(i+c)(i+c-1)x^{i+c-2} - 2x\sum_{i=0}^{\infty} a_i(i+c)x^{i+c-1} + n(n+1)\sum_{i=0}^{\infty} a_i x^{i+c} &= 0\\
\sum_{i=0}^{\infty} a_i(i+c)(i+c-1)x^{i+c-2} + \sum_{i=0}^{\infty} a_i\left(n(n+1) - (i+c)(i+c+1)\right)x^{i+c} &= 0\\
a_0c(c-1)x^{c-2} + a_1(c+1)cx^{c-1} + \sum_{i=2}^{\infty} a_i(i+c)(i+c-1)x^{i+c-2} &\\
+ \sum_{i=0}^{\infty} a_i\left(n(n+1) - (i+c)(i+c+1)\right)x^{i+c} &= 0\\
a_0c(c-1)x^{c-2} + a_1(c+1)cx^{c-1} + \sum_{i=0}^{\infty}\left(a_i(i+c)(i+c-1) + a_{i-2}(n(n+1) - (i+c-2)(i+c-1))\right) &= 0
\end{aligned}$$

It is clear that the indicial equation gives $c = 0$ or $c = 1$ and so the solution is found using Frobenius method 1 with the recurrence relation

$$a_i = \frac{(i-1)(i-2) - n(n+1)}{i(i-1)}a_{i-2}$$

This gives the first few terms as

$$\begin{aligned}
3i &= 2 & a_2 &= \frac{-n(n+1)}{2}a_0 & &\\
i &= 3 & a_3 &= \frac{2.1 - n(n+1)}{3.2}a_1 & &\\
i &= 4 & a_4 &= \frac{3.2 - n(n+1)}{4.3}a_2 & &= \frac{-(3.2 - n(n+1))n(n+1)}{4!}a_0\\
i &= 5 & a_5 &= \frac{4.3 - n(n+1)}{5.4}a_3 & &= \frac{(4.3 - n(n+1))(2.1 - n(n+1))}{5!}a_1
\end{aligned}$$

So the solution to Legendre's equation is

$$y = a_0\left(1 - \frac{n(n+1)}{2!}x^2 + \frac{-(3.2 - n(n+1))n(n+1)}{4!}x^4 + \dots\right) \\ + a_1\left(x + \frac{2.1 - n(n+1)}{3!}x^3 + \frac{(4.3 - n(n+1))(2.1 - n(n+1))}{5!}x^5 + \dots\right)$$

If $n \notin \mathbb{Z}$ then the two series are infinite and converge for $|x| < 1$. This kind of solution is of little further interest.

If $n \in \mathbb{Z}$ then one of the series terminates and so is a polynomial. These special polynomials are called Legendre polynomials, so the solution is of the form

$$y = AP_n(x) + BQ_n(x)$$

where P_n is the nth Legendre polynomial, and Q_n is a power series.

Having found one of the Legendre polynomials, the reduction of order method can be used to find a function for the power series.

$$\begin{aligned} &\text{If } n = 0 \quad P_0 = 1 && \text{and} \quad Q_0 = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) \\ &\text{If } n = 1 \quad P_1 = x && \text{and} \quad Q_1 = \frac{x}{2}\ln\left(\frac{1+x}{1-x}\right) - 1 \\ &\text{If } n = 2 \quad P_1 = \frac{1}{2}\left(3x^2 - 1\right) \end{aligned}$$

(14.2.2) Generating Legendre Polynomials

It is obvious to ask "what are the Legendre polynomials?" What is not so obvious, unfortunately, is actually finding them. The solution to Legendre's equation suggests that a generating function is

$$P_n(x) = \begin{cases} 1 - \frac{n(n+1)}{2!}x^2 + \frac{-(3.2-n(n+1))n(n+1)}{4!}x^4 + \dots & \text{if } n \text{ is even or zero} \\ x + \frac{2.1-n(n+1)}{3!}x^3 + \frac{(4.3-n(n+1))(2.1-n(n+1))}{5!}x^5 + \dots & \text{if } n \text{ is odd} \end{cases} \tag{26}$$

However, this is clearly is rather unwieldy and not particularly practical.

A better generating function is

$$\frac{1}{\sqrt{1 - 2xt + t^2}} = \sum_{n=0}^{\infty} P_n(x)t^n \tag{27}$$

Using a binomial expansion on the left hand side, then for sufficiently 'small' t,

$$\begin{aligned} \left(1 + \left(-2xt + t^2\right)\right)^{\frac{-1}{2}} &= 1 + \frac{-1}{2}(-2xt + t^2) + \frac{\frac{-1}{2}\frac{-3}{2}}{2!}(-2xt + t^2)^2 + \dots \\ &= 1 + xt + \frac{1}{2}(3x^2 - 1)t^2 + \dots \\ &= P_0 + tP_1 + t^2P_2 + \dots \end{aligned}$$

Of course, this is not a proof. Having seen the result in action, this is now formalised.

Theorem 28 *The Legendre polynomial generating function*

$$\frac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{\infty} P_n(x)t^n$$

i.e. equation (27) is correct.

Proof. First of all, differentiate equation (27) (partially) twice with respect to x, giving

$$t(1-2xt+t^2)^{\frac{-3}{2}} = \sum_{n=0}^{\infty} P_n' t^n \tag{29}$$

$$3t^2(1-2xt+t^2)^{\frac{-5}{2}} = \sum_{n=0}^{\infty} P_n'' t^n \tag{30}$$

The expressions on the right hand sides will be used to substitute for the left hand sides later. Now differentiate equation (27) (partially) with respect to t,

$$(x-t)(1-2xt+t^2)^{\frac{-3}{2}} = \sum_{n=0}^{\infty} P_n(x)nt^{n-1}$$

now multiply by t^2

$$t^2(x-t)(1-2xt+t^2)^{\frac{-3}{2}} = \sum_{n=0}^{\infty} P_n(x)nt^{n+1}$$

now differentiate with respect to t again

$$t^2\left(3(x-t)^2(1-2xt+t^2)^{\frac{-5}{2}} + (-1)(1-2xt+t^2)^{\frac{-3}{2}}\right) \tag{31}$$

$$+2t\left(2t(x-t)(1-2xt+t^2)^{\frac{-3}{2}}\right) = \sum_{n=0}^{\infty} P_n n(n+1)t^n$$

$$3t^2(x-t)^2\left(1-2xt+t^2\right)^{\frac{-5}{2}} + t(2x-3t)\left(1-2xt+t^2\right)^{\frac{-3}{2}} = \tag{32}$$

The idea is to show that equation (27) is a solution to Legendre's equation for all t. Hence consider the combination

$$(1-x^2)\sum_{n=1}^{\infty} P''(n)t^n - 2x\sum_{n=0}^{\infty} P_n'(x)t^n + \sum_{n=0}^{\infty} P_n(x)n(n+1)t^n$$

$$= (1-x^2)3t^2(1-2xt+t^2)^{\frac{-5}{2}} - 2xt(1-2xt+t^2)^{\frac{-3}{2}} + 3t^2(x-t)^2\left(1-2xt+t^2\right)^{\frac{-5}{2}} + t(2x-3t)\left(1-2xt+t^2\right)^{\frac{-3}{2}}$$

$$= -3t^2\left(1-2xt+t^2\right)^{\frac{-3}{2}} + 3t^2\left(1-2xt+t^2\right)^{\frac{-5}{2}} + 3t^2(-2xt+t^2)\left(1-2xt+t^2\right)^{\frac{-5}{2}}$$

$$= -3t^2\left(1-2xt+t^2\right)^{\frac{-3}{2}} + 3t^2\left(1-2xt+t^2\right)^{\frac{-5}{2}} + 3t^2(-2xt+t^2-1+1)\left(1-2xt+t^2\right)^{\frac{-5}{2}}$$

$$= +3t^2\left(1-2xt+t^2\right)^{\frac{-5}{2}} + 3t^2(-1)\left(1-2xt+t^2\right)^{\frac{-5}{2}}$$

$$= 0$$

Hence

$$\sum_{n=0}^{\infty}\left((1-x^2)P_n''(x) - 2xP_n'(x) + n(n+1)P_n(x)\right)t^n = 0$$

for all t. Since this must hold for all t clearly the coefficients must all be zero, so the $P_n(x)$ functions obey

Legendre's equation and so are Legendre polynomials. The conclusion, therefore, is that equation (27) is correct. □

Using The Generating Function

Since the generating function holds for all t, a value of t can be chosen at will and a relationship deduced. Other relationships can also be found.

Example 33 *By substituting $x = 1$ into the generating function, show that $P_n(1) = 1$.*

Proof. Solution For left hand side,

$$\begin{aligned}(1-2t-t^2)^{\frac{-1}{2}} &= (1-t)^{-1} \\ &= 1+t+t^2+\cdots = \sum_{n=0}^{\infty} t^n\end{aligned}$$

Comparing this with the right hand side,

$$\sum_{n=0}^{\infty} t^n = \sum_{n=0}^{\infty} P_n(1)t^n$$

So by comparing coefficients, it is clear that $P_n(1) = 0$. □

The generating function can also be used to deduce recurrence relations between Legendre polynomials.

Theorem 34 *Where P_n is the nth Legendre polynomial,*

$$(n+1)P_{n+1}(x) - (2n+1)xP_n(x) + nP_{n-1}(x) = 0$$

is a recurrence relation between three consecutive Legendre polynomials.

Proof. Starting with the generating function, equation (27),

$$\begin{aligned}(1-2xt+t^2)^{\frac{-1}{2}} &= \sum_{n=0}^{\infty} P_n(x)t^n \quad \text{equaiton(27). Differentiating w.r.t. } t, \\ (x-t)(1-2xt-t^2)^{\frac{-3}{2}} &= \sum_{n=0}^{\infty} P_n(x)nt^{n-1} \quad \text{now multiply by } (1-2xt+t^2) \\ (x-t)(1-2xt-t^2)^{\frac{-1}{2}} &= (1-2xt-t^2)\sum_{n=0}^{\infty} P_n(x)nt^{n-1} \quad \text{now use (27)} \\ (x-t)\sum_{n=0}^{\infty} P_n(x)t^n &= \sum_{n=0}^{\infty} P_n(x)nt^{n-1} - 2x\sum_{n=0}^{\infty} P_n(x)nt^n + \sum_{n=0}^{\infty} P_n(x)nt^{n+1} \\ x\sum_{n=0}^{\infty} P_n(x)t^n - \sum_{n=0}^{\infty} P_n(x)t^{n+1} &= \sum_{n=0}^{\infty} P_n(x)nt^{n-1} - 2x\sum_{n=0}^{\infty} P_n(x)nt^n + \sum_{n=0}^{\infty} P_n(x)nt^{n+1}\end{aligned}$$

Equating the coefficients of powers of t,

$$\begin{aligned}xP_n(x) - P_{n-1}(x) &= (n+1)P_{n+1}(x) - 2xnP_n(x) + (n-1)P_{n-1}(x) \\ 0 &= (n+1)P_{n+1}(x) - (2n+1)xP_n(x) + nP_{n-1}(x)\end{aligned}$$

Hence the result. □

This recurrence relation can be used to calculate the Legendre polynomials given that $P_0 = 1$ and $P_1 = x$. However, this is not a particularly efficient way to calculate them. A more efficient method would be to use Rodrigues' equation, which is

$$P_n(x) = \frac{1}{2^n n!}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left((x^2-1)^n\right) \tag{35}$$

This can be proved by induction.

(14.2.3) Orthogonality

Definition 36 *If f and g are appropriately integrable functions, then they are said to be orthogonal if $\int_x f(x)\,g(x)\,\mathrm{d}x = 0$ where the integration is taken over the appropriate interval. If the integral has value 1 then f and g are orthonormal.*

Rodrigues' formula is useful in showing the orthogonality of Legendre polynomials. First of all, note the following result.

$$\begin{aligned}
\int_{-1}^{1} f(x)P_n(x)\,\mathrm{d}x &= \frac{1}{2^n n!}\int_{-1}^{1} f(x)\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left((x^2-1)^n\right)\,\mathrm{d}x \quad \text{integrate by parts}\\
&= \frac{1}{2^n n!}\left[f(x)\frac{\mathrm{d}^{n-1}}{\mathrm{d}x^{n-1}}\left((x^2-1)^n\right)\right]_{-1}^{1} - \frac{1}{2^n n!}\int_{-1}^{1} f'(x)\frac{\mathrm{d}^{n-1}}{\mathrm{d}x^{n-1}}\left((x^2-1)^n\right)\,\mathrm{d}x\\
&= -\frac{1}{2^n n!}\int_{-1}^{1} f'(x)\frac{\mathrm{d}^{n-1}}{\mathrm{d}x^{n-1}}\left((x^2-1)^n\right)\,\mathrm{d}x
\end{aligned}$$

and repeating this process a further $n-1$ times it is evident that

$$= \frac{(-1)^n}{2^n n!}\int_{-1}^{1} f^{(n)}(x)(x^2-1)^n\,\mathrm{d}x$$

Now consider $f(x) = P_m(x)$. Then,

- when $m < n$, differentiating P_m n times will reduce it to zero. Hence $\int_{-1}^{1} P_m(x)P_n(x)\,\mathrm{d}x = 0$ for $m < n$.
- when $m > n$ by symmetry with the previous case, the integral is again zero.
- when $m = n$,

$$\begin{aligned}
\int_{-1}^{1} P_n(x)P_n(x)\,\mathrm{d}x &= \frac{(-1)^n}{2^n n!}\int_{-1}^{1}(x^2-1)^n P_n^{(n)}(x)\,\mathrm{d}x\\
&= \frac{(-1)^n}{2^{2n}(n!)^2}\int_{-1}^{1}(x^2-1)^n\frac{\mathrm{d}^{2n}}{\mathrm{d}x^{2n}}(x^2-1)^n\,\mathrm{d}x
\end{aligned}$$

but $(x^2-1)^n = x^{2n} - nx^{n-1} + \cdots + (-1)^n$ and differentiating this $2n$ times will make all but the first term disappear, leaving $(2n)!$. Hence

$$= \frac{(-1)^n(2n)!}{2^{2n}(n!)^2}\int_{-1}^{1}(x^2-1)^n\,\mathrm{d}x$$

This integral can be evaluated by means of a reduction formula by putting $x = \sin\theta$. It turns out that

$$\int_{-1}^{1} P_n^2\,\mathrm{d}x = \frac{2}{2n+1}$$

In general,

$$\int_{-1}^{1} P_m(x)P_n(x) = \frac{2}{2n+1}\delta_{m,n}$$

where

$$\delta_{m,n} = \begin{cases} 1 & \text{if } m = n \\ 0 & \text{if } m \neq n \end{cases}$$

is called the Kronecker delta.

(14.2.4) Fourier-Legendre Expansion

The orthogonality of the sine and cosine functions allows Fourier expansion. Since Legendre polynomials are orthogonal, it seems reasonable that a function should be able to be expanded in terms of them, say

$$f(x) = a_0P_0(x) + a_1P_1(x) + \cdots = \sum_{n=0}^{\infty} a_nP_n(x)$$

Consider 'resolving' f in the 'direction' of each Legendre polynomial by taking the inner product

$$\begin{aligned}
\int_{-1}^{1} f(x)P_m(x)\,\mathrm{d}x &= \int_{-1}^{1} P_m(x)\sum_{n=0}^{\infty} a_nP_n(x)\,\mathrm{d}x \\
&= \sum_{n=0}^{\infty} a_n\left(\int_{-1}^{1} P_n(x)P_m(x)\,\mathrm{d}x\right) \quad \text{since the integral and the sum converge} \\
&= \sum_{n=0}^{\infty} a_n\frac{2\delta_{m,n}}{2m+1} \\
&= \frac{2a_m}{2m+1}
\end{aligned}$$

Hence having found the coefficients,

$$f(x) = \sum_{n=0}^{\infty} P_n(x)\left(\frac{2n+1}{2}\int_{-1}^{1} f(x)P_n(x)\,\mathrm{d}x\right) \tag{37}$$

This is called the Fourier-Legendre expansion of a function.

Note that a finite polynomial has a finite Fourier-Legendre expansion.

(14.3) Bessel Functions

Legendre polynomials are one particular kind of solution to differential equations, especially where spherical symmetry is concerned. Another such class of functions are the Bessel Functions, which are common in situations with cylindrical symmetry. They also have uses in evaluating functions of the form $\sin\sin x$.

It is convenient at this point to draw attention to the Gamma function, as seen in Chapter **??** and elsewhere. It is defined by

$$\Gamma(n) = \int_{0}^{\infty} e^{-x}x^{n-1}\,\mathrm{d}x \qquad n > 0 \tag{38}$$

A recurrence relation is now sought by considering

$$\begin{aligned}
\Gamma(n+1) &= \int_0^\infty e^{-x}x^n \, \mathrm{d}x \\
\text{parts} \quad & \begin{matrix} u = x^n & \frac{\mathrm{d}v}{\mathrm{d}x} = e^{-x} \\ \frac{\mathrm{d}u}{\mathrm{d}x} = nx^{n-1} & v = -e^{-x} \end{matrix} \\
&= \left[-x^n e^{-x}\right]_0^\infty + \int_0^\infty ne^{-x}x^{n-1} \, \mathrm{d}x \\
&= n\int_{n=0}^\infty e^{-x}x^{n-1} \, \mathrm{d}x = n\Gamma(n)
\end{aligned}$$

By evaluating the appropriate integral, $\Gamma(1) = 1$ and so by induction it is clear that $\Gamma(n+1) = n!$. Furthermore, this can be used to give meaning to factorials of non-integer numbers since, for example, $\Gamma\left(\frac{3}{2}\right) = \frac{1}{2}\Gamma\left(\frac{1}{2}\right)$. Now,

$$\begin{aligned}
\Gamma\left(\frac{1}{2}\right) &= \int_0^\infty e^{-x}x^{\frac{-1}{2}} \, \mathrm{d}x \qquad \text{now put } x = u^2 \text{ so } \frac{\mathrm{d}x}{\mathrm{d}u} = 2u \text{ to give} \\
&= 2\int_0^\infty e^{-u^2}\frac{u}{u} \, \mathrm{d}u = 2\int_0^\infty e^{-u^2} \, \mathrm{d}u \\
\left(\Gamma\left(\frac{1}{2}\right)\right)^2 &= \left(2\int_0^\infty e^{-u^2} \, \mathrm{d}(u)\right)\left(2\int_0^\infty e^{-v^2} \, \mathrm{d}v\right) \\
&= 4\int_0^\infty \int_0^\infty e^{-(u^2+v^2)} \, \mathrm{d}u \, \mathrm{d}v \qquad \text{since the limits are independent} \\
&= 4\int_{\theta=0}^{\frac{\pi}{2}} \int_{r=0}^\infty re^{-r^2} \, \mathrm{d}r \, \mathrm{d}\theta \qquad \text{by changing to polar co-ordinates} \\
&= 4\int_0^{\frac{\pi}{2}} \left[\frac{-1}{2}e^{-r^2}\right]_0^\infty \mathrm{d}\theta \\
&= 4\int_0^{\frac{\pi}{2}} \frac{1}{2} \, \mathrm{d}\theta \\
&= \pi \\
\text{hence} \quad \Gamma\left(\frac{1}{2}\right) &= \sqrt{\pi}
\end{aligned}$$

Hence an expression for any half integer factorial can be found. Note that this can be extended to negative values, since by rearranging the recurrence relation, $\Gamma(n) = \dfrac{\Gamma(n+1)}{n}$.

Finally in this preamble to Bessel functions, note the Pockhammer symbol — a subscript — used to express factorial-like expressions,

$$(\alpha)_r = \alpha(\alpha+1)(\alpha+2)\dots(\alpha+(r-1)) \tag{39}$$

Notice that $(1)_n = n!$.

(14.3.1) Bessel's Equation And Its Solutions

Bessel's equation is

$$x^2\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + x\frac{\mathrm{d}y}{\mathrm{d}x} + (x^2 - \nu^2)y = 0 \quad \nu \in \mathbb{R} \tag{40}$$

Clearly $x = 0$ is a singular point. However,

$$x\frac{x}{x^2} = 1 \qquad x^2\frac{x^2-\nu^2}{x^2} = x^2 - \nu^2$$

so these are regular singular points with infinite radii of convergence. The Frobenius solutions will therefore hold everywhere. Frobenius method works through as follows.

$$x^2 \sum_{n=0}^{\infty} (n+c)(n+c-1)a_n x^{n+c-2} + x \sum_{n=0}^{\infty} (n+c)a_n x^{n+c-1} + (x^2 - \nu^2) \sum_{n=0}^{\infty} a_n x^{n+c} = 0$$

$$\sum_{n=0}^{\infty} \Big((n+c)(n+c-1) + (n+c) - \nu^2 \Big) a_n c^{n+c} + \sum_{n=0}^{\infty} a_n x^{n+c+2} = 0$$

$$a_0(c^2 - \nu^2)x^c + a_1 \Big((1+c)^2 - \nu^2 \Big) x^{c+1} \sum_{n=2}^{\infty} \Big((n+c)^2 - \nu^2 \Big) a_n x^{n+c} + \sum_{n=0}^{\infty} a_n x^{n+c+2} = 0 \tag{41}$$

$$a_0(c^2 - \nu^2)x^c + a_1 \Big((1+c)^2 - \nu^2 \Big) x^{c+1} \sum_{n=2}^{\infty} \Big(\Big((n+c)^2 - \nu^2 \Big) a_n + a_{n-2} \Big) x^{n+c} = 0 \tag{42}$$

So the indicial equation is $c^2 - \nu^2 = 0$ so that $c = \pm\nu$. The Frobenius method to use from this point is dependent upon the value of the parameter ν and so the three different cases are considered in turn.

Case 1. The simplest case is when $2\nu \notin \mathbb{Z}$ (since $\nu - (-\nu) = 2\nu$) so that each value of c will produce a linearly independent solution.

Consider $c = \nu$ then from equation 42 the recurrence relation is

$$a_n = \frac{-a_{n-2}}{(n+\nu)^2 - \nu^2} = \frac{-a_{n-2}}{n(n+2\nu)}$$

Note that in this case $a_1 = 0$, and the coefficients are found as

$$\begin{aligned} a_2 &= \frac{-a_0}{2(n+2\nu)} \\ a_4 &= \frac{-a_2}{4(4+2\nu)} = \frac{a_0}{4.2(4+2\nu)(2+2\nu)} \\ \text{so} \quad a_{2n} &= \frac{(-1)^n a_0}{(2^n n!)(2^n(1+\nu)_n} \end{aligned}$$

So one of the solutions is

$$\begin{aligned} y &= a_0 x^\nu \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n} n! (1+\nu)_n} \\ &= \frac{A x^\nu}{2^\nu \Gamma(1+\nu)} \sum_{n=0}^{\infty} \frac{\left(\frac{x^2}{4}\right)^n}{n!(1+\nu)_n} \quad \text{by putting } A = a_0 2^\nu \Gamma(1+\nu) \\ &= A J_\nu(x) \end{aligned}$$

The function $J_\nu(x)$ is the Bessel function of order ν and is given by

$$J_\nu(x) = \frac{x^\nu}{2^\nu \Gamma(1+\nu)} \sum_{n=0}^{\infty} \frac{\left(\frac{-x^2}{4}\right)^n}{n!(1+\nu)_n} \tag{43}$$

The other value $c = -\nu$ produces the other Bessel function—of order $-\nu$—which is

$$J_{-\nu}(x) = \frac{x^{-\nu}}{2^{-\nu} \Gamma(1-\nu)} \sum_{n=0}^{\infty} \frac{\left(\frac{-x^2}{4}\right)^n}{n!(1-\nu)_n} \tag{44}$$

The general solution to Bessel's equation—equation 40— in the case where $2\nu \notin \mathbb{Z}$ is quite simply $y = AJ_\nu(x) + BJ_{-\nu}(x)$.

Case 2. If $\nu = 0$ then the indicial equation has a repeated root. Bessel's equation becomes

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + \frac{1}{x}\frac{\mathrm{d}y}{\mathrm{d}x} + y = 0$$

It can be shown that the solution is of the form $y = AJ_0(x) + BY_0(x)$ where Y_0 is Weber's Bessel function of order 0. This is given by

$$Y_0(x) = J_0(x)\ln x - \sum_{n=0}^{\infty} \frac{\phi(n)}{(n!)^2}\left(\frac{-x^2}{4}\right)^n \quad \text{where} \quad \phi(n) = \sum_{i=1}^{n}\frac{1}{i}$$

This equation is found by using Frobenius General Rule 3.

Case 3. When $\nu \in \mathbb{Z}$ the solution is of the form $y = AJ_\nu(x) + BY_\nu(x)$.

Generating Functions For Bessel Equations

As with Legendre polynomials, Bessel functions can be generated in a convenient way. First of all it is convenient to express Bessel functions as being contained in a sum.

$$\begin{aligned} J_n(x) &= \frac{x^n}{2^n\Gamma(1+n)}\sum_{i=0}^{\infty}\frac{\left(\frac{-x^2}{4}\right)^i}{i!(1+n)_i} \\ &= \sum_{i=0}^{\infty}\frac{(-1)^i x^{2i+n}}{2^{2i+n}i!(1+n)_i\Gamma(1+n)} \\ &= \sum_{i=0}^{\infty}\frac{(-1)^i x^{2i+n}}{2^{2i+n}i!(n+i)!} \end{aligned} \tag{45}$$

By considering the series expansion of $e^{\frac{1}{2}x\left(t-\frac{1}{t}\right)} = e^{\frac{1}{2}xt}e^{-\frac{x}{2t}}$ it can be shown that

$$e^{\frac{1}{2}x\left(t-\frac{1}{t}\right)} = \sum_{n=-\infty}^{\infty} J_n(x)t^n$$

Notice the left hand side means that the generating function does not change under the mapping $t \mapsto \frac{-1}{t}$. Using this on the right hand side,

$$\begin{aligned} \sum_{n=-\infty}^{\infty} J_n(x)t^n &= \sum_{n=-\infty}^{\infty} J_n(x)\left(\frac{-1}{t}\right)^n \\ &= \sum_{m=-\infty}^{\infty} J_{-m}(x)(-1)^m t^m \quad \text{by putting } n = -m \\ \text{comparing terms,} \quad J_n(x) &= (-1)^n J_{-n}(x) \end{aligned}$$

This means that $J_n(x)$ and $J_{-n}(x)$ are linearly dependent, which is consistent with them not being the two solutions to Bessel's equation.

The purpose of a generating function is to find relationships such as these. However, it is possible to deduce two important results without the use of the generating function. From (45),

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\left(x^{\nu}J_{\nu}(x)\right) &= \frac{\mathrm{d}}{\mathrm{d}x}\left(\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+2\nu}}{2^{2n+\nu}n!\Gamma(1+\nu+n)}\right)\\
&= \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+2\nu-1}(2n+2\nu)}{2^{2n+\nu}n!\Gamma(1+\nu+n)}\\
&= \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+2\nu-1}}{2^{2n+\nu-1}n!\Gamma(\nu+n)} \quad \text{since } \Gamma(1+\nu+n)=(\nu+n)!\\
&= x^{\nu}\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+(\nu-1)}}{2^{2n+(\nu-1)}n!\Gamma(1+(\nu-1)+n)}\\
&= x^{\nu}J_{\nu-1}(x)
\end{aligned}
$$

Also,

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\left(x^{-\nu}J_{\nu}(x)\right) &= \frac{\mathrm{d}}{\mathrm{d}x}\left(\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{2^{2n+\nu}n!\Gamma(1+\nu+n)}\right)\\
&= \sum_{n=0}^{\infty}\frac{(-1)^n 2n x^{2n-1}}{2^{2n+\nu}n!\Gamma(1+\nu+n)} = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n-1}}{2^{2n+\nu-1}(n-1)!\Gamma(1+\nu+n)}\\
&= \sum_{n=1}^{\infty}\frac{(-1)^n x^{2n-1}}{2^{2n+\nu-1}(n-1)!\Gamma(1+\nu+n)} \quad \text{due to the factor of } n \text{ in the numerator.}\\
&= \sum_{m=0}^{\infty}\frac{(-1)^{m+1}x^{2m+1}}{2^{2m+\nu+1}(m)!\Gamma(1+\nu+(m+1))}\\
&= \frac{1}{x^{\nu}}\sum_{m=0}^{\infty}\frac{(-1)^{m+1}x^{2m+(\nu+1)}}{2^{2m+(\nu+1)}(m)!\Gamma(1+(\nu+1)+m)}\\
&= -x^{-\nu}J_{\nu+1}(x)
\end{aligned}
$$

The two above results have the obvious use of establishing a recurrence relationship.

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\left(x^{\nu}J_{\nu}(x)\right) &= x^{\nu}J_{\nu-1}(x)\\
\nu x^{\nu-1}J_{\nu}(x)+x^{\nu}J'_{\nu}(x) &=\\
J'_{\nu}(x) &= J_{\nu-1}(x)-\frac{\nu}{x}J_{\nu}(x)\\
\text{and also}\quad \frac{\mathrm{d}}{\mathrm{d}x}\left(x^{-\nu}J_{\nu}(x)\right) &= -x^{\nu}J_{\nu+1}(x)\\
-\nu x^{-\nu-1}J_{\nu}(x)+x^{-\nu}J'_{\nu}(x) &=\\
J'_{\nu}(x) &= -J_{\nu+1}(x)+\frac{\nu}{x}J_{\nu}(x)
\end{aligned}
$$

Using these two equations to eliminate $J'_{\nu}(x)$ gives

$$
\begin{aligned}
J_{\nu-1}(x)-\frac{\nu}{x}J_{\nu}(x) &= -J_{\nu+1}(x)+\frac{\nu}{x}J_{\nu}(x)\\
J_{\nu+1}(x) &= \frac{2\nu}{x}J_{\nu}(x)-J_{\nu-1}(x)
\end{aligned}
$$

Orthogonality & Bessel Fourier Series

As has been seen, Bessel functions are not orthogonal. However, because of the irregular nature of their roots, it is now shown that $J_n(\lambda x)$ and $J_n(\mu x)$ where $\mu, \lambda \in \mathbb{R}$ are orthogonal. Now, since $J_n(x)$ is a Bessel function it satisfied Bessel's equation,

$$x^2 \frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(J_n(x)\right) + x\frac{\mathrm{d}}{\mathrm{d}x}\left(J_n(x)\right) + \left(x^2 - n^2\right) J_n(x) = 0$$

Now replace x with λx to give

$$\lambda^2 x^2 \frac{\mathrm{d}^2}{\mathrm{d}(\lambda x)^2}\left(J_n(\lambda x)\right) + \lambda x\frac{\mathrm{d}}{\mathrm{d}\lambda x}\left(J_n(\lambda x)\right) + \left(\lambda^2 x^2 - n^2\right) J_n(\lambda x) = 0$$

By the chain rule, $\frac{\mathrm{d}}{\mathrm{d}x} = \frac{\mathrm{d}(\lambda x)}{\mathrm{d}x}\frac{\mathrm{d}}{\mathrm{d}(\lambda x)}$ and so $\frac{\mathrm{d}}{\mathrm{d}(\lambda x)} = \frac{1}{\lambda}\frac{\mathrm{d}}{\mathrm{d}x}$. Hence

$$x^2 \frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(J_n(\lambda x)\right) + x\frac{\mathrm{d}}{\mathrm{d}x}\left(J_n(\lambda x)\right) + \left(\lambda^2 x^2 - n^2\right) J_n(\lambda x) = 0$$

Since this is zero, multiplying by $\frac{J_n(\mu x)}{x}$ and integrating between 0 and a gives

$$\int_0^a J_n(\mu x)\left(x\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(J_n(\lambda x)\right) + \frac{\mathrm{d}}{\mathrm{d}x}\left(J_n(\lambda x)\right) + \frac{1}{x}\left(\lambda^2 x^2 - n^2\right) J_n(\lambda x)\right)\,\mathrm{d}x = 0$$

and now by reversing the product rule,

$$\int_0^a J_n(\mu x)\left(\frac{\mathrm{d}}{\mathrm{d}x}\left(x\frac{\mathrm{d}}{\mathrm{d}x}\left(J_n(\lambda x)\right)\right) + \frac{1}{x}\left(\lambda^2 x^2 - n^2\right) J_n(\lambda x)\right)\,\mathrm{d}x = 0$$

But exactly the same can be done starting with $J_n(\mu x)$ instead of $J_n(\lambda x)$. Therefore μ and λ can be 'interchanged' to give

$$\int_0^a J_n(\lambda x)\left(\frac{\mathrm{d}}{\mathrm{d}x}\left(x\frac{\mathrm{d}}{\mathrm{d}x}\left(J_n(\mu x)\right)\right) + \frac{1}{x}\left(\mu^2 x^2 - n^2\right) J_n(\mu x)\right)\,\mathrm{d}x = 0$$

now subtracting the two above equations,

$$\int_0^a J_n(\mu x)\frac{\mathrm{d}}{\mathrm{d}x}\left(x\frac{\mathrm{d}}{\mathrm{d}x}\left(J_n(\lambda x)\right)\right) - J_n(\lambda x)\frac{\mathrm{d}}{\mathrm{d}x}\left(x\frac{\mathrm{d}}{\mathrm{d}x}\left(J_n(\mu x)\right)\right) + xJ_n(\lambda x)J_n(\mu x)\left(\lambda^2 - \mu^2\right)\,\mathrm{d}x \tag{46}$$

Now, the first two terms of the integrand can be easily integrated by parts. Integrating each term,

$$\int_0^a J_n(\mu x)\frac{\mathrm{d}}{\mathrm{d}x}\left(x\frac{\mathrm{d}}{\mathrm{d}x}\left(J_n(\lambda x)\right)\right)\,\mathrm{d}x = \left[J_n(\mu x)x\frac{\mathrm{d}}{\mathrm{d}x}\left(J_n(\lambda x)\right)\right]_0^a - \int_0^a x\frac{\mathrm{d}}{\mathrm{d}x}\left(J_n(\lambda x)\right)x\frac{\mathrm{d}}{\mathrm{d}x}\left(J_n(\mu x)\right)\,\mathrm{d}x$$
$$\int_0^a J_n(\lambda x)\frac{\mathrm{d}}{\mathrm{d}x}\left(x\frac{\mathrm{d}}{\mathrm{d}x}\left(J_n(\mu x)\right)\right)\,\mathrm{d}x = \left[J_n(\lambda x)x\frac{\mathrm{d}}{\mathrm{d}x}\left(J_n(\mu x)\right)\right]_0^a - \int_0^a x\frac{\mathrm{d}}{\mathrm{d}x}\left(J_n(\mu x)\right)x\frac{\mathrm{d}}{\mathrm{d}x}\left(J_n(\lambda x)\right)\,\mathrm{d}x$$

This leaves the final term, which is of the form of a definition of orthogonality with a surplus factor and a weighting function $p(x) = x$. Going back to equation (46)

$$\begin{aligned}(\lambda^2 - \mu^2)\int_0^a xJ_n(\lambda x)J_n(\mu x)\,\mathrm{d}x &= \left[J_n(\mu x)x\frac{\mathrm{d}}{\mathrm{d}x}\left(J_n(\lambda x)\right)\right]_0^a - \left[J_n(\lambda x)x\frac{\mathrm{d}}{\mathrm{d}x}\left(J_n(\mu x)\right)\right]_0^a \\ &= J_n(\mu a)a\frac{\mathrm{d}}{\mathrm{d}x}\left(J_n(\lambda a)\right) - J_n(\lambda a)a\frac{\mathrm{d}}{\mathrm{d}x}\left(J_n(\mu a)\right)\end{aligned}$$

Choosing λ and μ so that λa and μa are distinct roots of $J_n(x)$ produces

$$\int_0^a xJ_n(\lambda x)J_n(\mu x)\,\mathrm{d}x = 0$$

Hence the orthogonality is shown. Notice the presence of the weighting function ensures that the integrand has value 0 at the origin. The choices of μ and λ also mean that the functions also have value 0 at $x = a$.

Having found a kind of orthogonality in Bessel functions, it is now of interest as to how to find a Fourier-Bessel expansion, say

$$f(x) = \sum_{i=1}^{\infty} c_i J_n(\lambda_i x) \quad \text{where } J_n(\lambda_i a) = 0$$

In order to do this, evaluate the following integral

$$\int_0^a x J_n(\lambda_j x) f(x) \, \mathrm{d}x = \int_0^a x J_n(\lambda_j x) \sum_{i=1}^{\infty} c_i J_n(\lambda_i x) \, \mathrm{d}x$$

which is zero whenever $j \neq i$. This integral therefore 'picks out' a certain term in the sum, and so it follows that

$$\begin{aligned} &= \int_0^a x \left(J_n(\lambda_j x)\right)^2 c_j \, \mathrm{d}x \\ &= c_j \frac{\lambda_j a^2}{2} \left(J_n'(\lambda_j a)\right)^2 \quad \text{it can be shown.} \\ \text{hence} \quad c_j &= \frac{2}{\lambda_j a^2 \left(J_n'(\lambda_j a)\right)^2} \int_0^a a f(x) J_n \lambda_j x \, \mathrm{d}x \end{aligned}$$

(14.4) Transform Methods

(14.4.1) Laplace Transforms

De�nition & Basic Transforms

A transform allows differential and integral equations to be solved algebraically. The substitution $y = e^{mx}$ is an example of this for simple differential equations.

Definition 47 *The Laplace transform of a function f is $\mathcal{L}(f) = \int_0^{\infty} e^{-st} f(t) \, \mathrm{d}t$ which is a function of s.*

The existence of Laplace transforms is subject to the existence of this improper integral.

Definition 48 *A function $f(t)$ has exponential order if there exist constants A and b such that $|f(t)| < Ae^{bt}$ for $t \in \mathbb{R}_0^+$.*

Theorem 49 *A piecewise function of exponential order has a Laplace transform.*

Proof. From the definition of a Laplace transform,

$$\begin{aligned}
|\mathcal{L}(f)| &= \left|\int_0^\infty e^{-st} f(t)\, \mathrm{d}t\right| \\
&\leqslant \int_0^\infty \left|e^{-st} f(t)\right| \, \mathrm{d}t \quad \text{by the trinagle inequality} \\
&= \int_0^\infty \left|e^{-st}\right| |f(t)| \, \mathrm{d}t \\
&\leqslant \int_0^\infty e^{-st} A e^{bt} \, \mathrm{d}t \\
&= \int_0^\infty A e^{(b-s)t} \, \mathrm{d}t \\
&= \lim_{N\to\infty} \left(\frac{Ae^{(b-s)N}}{b-s}\right) - \frac{A}{b-s} \\
|\mathcal{L}(f)| &\leqslant \frac{A}{s-b} \quad \text{provided } s > b
\end{aligned}$$

Hence the result. □

Since in fact $0 \leqslant |\mathcal{L}(f)| \leqslant \frac{A}{s-b}$ this shows that a Laplace transform must have limit 0 as $s \to \infty$.

A few common Laplace transforms are as follows.

1. For $f(t) = e^{kt}$ where $k \in \mathbb{R}$,

$$\int_0^\infty e^{-st} e^{kt} \, \mathrm{d}t = \int_0^\infty e^{(k-s)t} \, \mathrm{d}t = \left[\frac{e^{(k-s)t}}{k-s}\right]_0^\infty = \frac{1}{s-k}$$

provided that $s > k$.

2. For $f(t) = cos(\omega t)$ use the exponential form to give

$$\mathcal{L}(\cos \omega t) = \frac{1}{2}\mathcal{L}\left(e^{i\omega t}\right) + \frac{1}{2}\mathcal{L}\left(e^{-i\omega t}\right) = \frac{1}{2}\left(\frac{1}{s-i\omega} + \frac{1}{s+i\omega}\right) = \frac{s}{s^2+\omega^2}$$

In this case k $(i\omega)$ is imaginary. A more correct statement of the condition for a Laplace transform to exist is that $\mathrm{Re}(s) > \mathrm{Re}(k)$ which is trivially satisfied here.

3. For $f(t) = \sin(\omega t)$ the calculation is similar to those above.

$$\mathcal{L}(\sin \omega t) = \frac{1}{2i}\mathcal{L}\left(e^{i\omega t}\right) - \frac{1}{2i}\mathcal{L}\left(e^{-i\omega t}\right) = \frac{1}{2i}\left(\frac{1}{s-i\omega} - \frac{1}{s+i\omega}\right) = \frac{\omega}{s^2+\omega^2}$$

4. For t^n where $n \in \mathbb{N}$

$$\mathcal{L}(t^n) = \int_0^\infty e^{-st} t^n \, \mathrm{d}t$$

Now put $x = st$ so $\dfrac{\mathrm{d}x}{\mathrm{d}t} = s$ and hence

$$\mathcal{L}(t^n) = \int_0^\infty e^{-x} \left(\frac{x}{s}\right)^n \frac{1}{s} \, \mathrm{d}t = \frac{1}{s^{n+1}} \int_0^\infty e^{-x} x^n \, \mathrm{d}x = \frac{\Gamma(n+1)}{s^{n+1}} = \frac{n!}{s^{n+1}}$$

In the cases of $\cos \omega t$ and $\sin \omega t$ it was assumed that the Laplace transform is a linear operator. This follows readily from its definition as an integral.

Shifting Theorems

Inverse Laplace transforms can be found using a complex contour integral, which is beyond the scope of this Chapter. An alternative method is to manipulate the expression to be inverse transformed into the form of a known Laplace transform. To this end, the following theorems are of great use.

Theorem 50 (First Shifting Theorem) *Suppose* $\mathcal{L}\left(f(t)\right) = F(s)$*, then* $\mathcal{L}\left(e^{at}f(t)\right) = F(s-a)$.

Proof. From the definition of a Laplace transform,

$$\mathcal{L}\left(e^{at}f(t)\right) = \int_0^\infty e^{-(s-a)t} f(t)\,\mathrm{d}t$$

But s is just some parameter, which here has been replaced by $s-a$. Hence the result. □

Theorem 51 (Second Shifting Theorem) *Suppose that a function* $f(t)$ *has Laplace transform* $F(s)$*. Then the function*

$$g(t) = \begin{cases} 0 & \text{if } t < a \\ f(t-a) & \text{if } t > a \end{cases}$$

has Laplace transform $e^{-sa}F(s)$

Proof. Evaluating the appropriate transform,

$$\begin{aligned}
\mathcal{L}\left(g\right) &= \int_0^\infty e^{-st} g(t)\,\mathrm{d}t \\
&= \int_a^\infty e^{-st} f(t-a)\,\mathrm{d}t \quad \text{now put } \tau = t-a \text{ so } \frac{\mathrm{d}\tau}{\mathrm{d}t} = 1 \\
&= \int_0^\infty e^{-s(\tau+a)} f(\tau)\,\mathrm{d}\tau \\
&= e^{-sa}\int_0^\infty e^{-s\tau} f(\tau)\,\mathrm{d}\tau
\end{aligned}$$

Hence the result. □

Notice that the first shifting theorem shifts in s while the second shifts in t.

Differential Equations

In order to solve differential equations using Laplace transforms it is necessary to find the transform of a derivative.

$$\begin{aligned}
\mathcal{L}\left(f'(t)\right) &= \int_0^\infty e^{-st} f'(t)\,\mathrm{d}t \quad \text{now integrate by parts} \\
&= \left[e^{-st}f(t)\right]_0^\infty + s\int_0^\infty e^{-st} f(t)\,\mathrm{d}t \\
&= s\mathcal{L}\left(f\right) - f(0)
\end{aligned}$$

For the second derivative it would be possible to use integration by parts twice. However, an easier way is to use the result for the first derivative.

$$\begin{aligned}
\mathcal{L}\left(f''\right) &= s\mathcal{L}\left(f'\right) - f'(0) \\
&= s\left(s\mathcal{L}\left(f\right) - f(0)\right) - f'(0) \\
&= s^2
\end{aligned}$$

$$Ltranf - sf(0) - f'(0)$$

In a similar way it can be shown that

$$\mathcal{L}\left(f'''\right) = s^3\mathcal{L}\left(f\right) - s^2 f(0) - sf'(0) - f''(0)$$

Using these results to substitute into a differential equation—not forgetting the inhomogeneous term—a solution can be found algebraically which must then be inverse transformed.

Convolution & Integral Equations

As differential equations are equations in terms of derivatives, integral equations are equations in terms of integrals. Beforehand it is necessary to develop more theory.

One way to find inverse transforms is to use partial fractions to get an expression in terms of transforms of e^{kt}. However, there is an alternative.

Definition 52 *Define the binary operation '$*$' as the convolution of two function f and g by*

$$f * g = \int_0^t f(\tau)g(t-\tau)\,\mathrm{d}\tau$$

Theorem 53 *Where '$*$' denotes convolution of functions, $\mathcal{L}\left(f*g\right) = \mathcal{L}\left(f\right)\mathcal{L}\left(g\right)$.*

Proof. By the definition of the Laplace transform

$$\begin{aligned}
\mathcal{L}\left(f*g\right) &= \int_0^\infty e^{-st}\int_0^t f(\tau)g(t-\tau)\,\mathrm{d}\tau\,\mathrm{d}t \\
&= \int_0^\infty \int_0^t e^{-st} f(\tau)g(t-\tau)\,\mathrm{d}\tau\,\mathrm{d}t
\end{aligned}$$

Now, notice that $\tau = 0$ to t and then $t = 0$ to ∞ is a triangular region as shown in Figure 10. The order of integration can therefore be changed to give

$$= \int_{\tau=0}^\infty \int_{t=\tau}^\infty e^{-st} f(\tau)g(t-\tau)\,\mathrm{d}t\,\mathrm{d}\tau$$

Now put $z = t-\tau$ giving $\dfrac{\mathrm{d}z}{\mathrm{d}t} = 1$

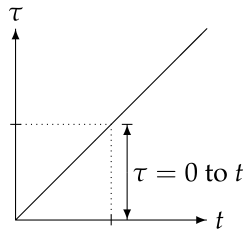
$$\begin{aligned}
&= \int_{\tau=0}^\infty f(\tau)\int_{z=0}^\infty e^{-s(z+\tau)} g(z)\,\mathrm{d}z\,\mathrm{d}\tau \\
&= \left(\int_{\tau=0}^\infty e^{-s\tau} f(\tau)\,\mathrm{d}\tau\right)\left(\int_{z=0}^\infty e^{-sz} g(z)\,\mathrm{d}z\right) \\
&= \mathcal{L}\left(f\right)\mathcal{L}\left(g\right)
\end{aligned}$$

Hence the result.

This convolution result allows Volterra Integral Equations to be solved. Not surprisingly these are of the form

$$y(t) = f(t) + \int_0^t y(\tau)K(t-\tau)\,\mathrm{d}\tau$$

where K is called the kernel of the equation. The equation can be written as $y = f + y*K$ from which the Laplace transform can be taken and a solution found.



□

Figure 1: Domain of integration for the Laplace transform of a convolution

(14.4.2) Generalised Functions

Common Generalised Functions

Generalised functions are peculiar things which are defined in terms of their action on an integral—they do not have a 'normal' algebraic definition.

Definition 54 *For a continuous function g,*

- *g is a good function if it decays rapidly as $x \to \pm\infty$ e.g. e^{-x}.*
- *g is a fairly good function if it decays algebraically as $x \to \pm\infty$ e.g. $\frac{1}{x}$.*

Definition 55 *Where g is a good function*

- *The unit function, $I(x)$ is defined by $\int_{-\infty}^{\infty} I(x)g(x)\,\mathrm{d}x = \int_{-\infty}^{\infty} g(x)\,\mathrm{d}x$. This function may be thought of as having value 1 everywhere except at a countable number of points.*
- *The Heaviside* function is defined by $\int_{-\infty}^{\infty} H(x)g(x)\,\mathrm{d}x = \int_{0}^{\infty} g(x)\,\mathrm{d}x$. The function may be thought of as the step function given by*

$$H(x) = \begin{cases} 0 & x < 0 \\ 1 & x > 0 \end{cases}$$

- *The Dirac delta function is defined by $\displaystyle\int_{-\infty}^{\infty} \delta(x)g(x) = g(0)$. No ordinary function can be equal to the Dirac delta function, but sequences of them can do. For example $f_n(x) = \dfrac{\sin nx}{\pi x}$.*
- *The signum function is defined by $\displaystyle\int_{-\infty}^{\infty} \operatorname{sgn}(x)g(x)\,\mathrm{d}x = \int_{0}^{\infty} g(x)\,\mathrm{d}x - \int_{-\infty}^{0} g(x)\,\mathrm{d}x$. This function can be expressed as*

$$\operatorname{sgn}(x) = \begin{cases} -1 & x < 0 \\ 1 & x > 0 \end{cases}$$

Although the above definitions seem rather diverse, there are in fact a number of relationships between generalised functions.

$$\begin{aligned} \int_{-\infty}^{\infty} (2H(x) - I(x))\, g(x)\,\mathrm{d}x &= 2\int_{-\infty}^{\infty} H(x)g(x)\,\mathrm{d}x - \int_{-\infty}^{\infty} I(x)g(x)\,\mathrm{d}x \\ &= 2\int_{0}^{\infty} g(x)\,\mathrm{d}x - \int_{-\infty}^{\infty} g(x)\,\mathrm{d}x \\ &= \int_{0}^{\infty} g(x)\,\mathrm{d}x - \int_{-\infty}^{0} g(x)\,\mathrm{d}x \\ &= \operatorname{sgn} x \end{aligned}$$

*Oliver Heaviside (1850–1925) was an English electrical engineer.

So $2H(x) - I(x) = \operatorname{sgn} x$.

Another useful result is

$$f(x)\delta(x) = f(0)\delta(x) \tag{56}$$

This is shown as follows,

$$\text{firstly, } \int_{-\infty}^{\infty} f(x)\delta(x)g(x)\,\mathrm{d}x = f(0)g(0)$$
$$\text{and secondly } \int_{-\infty}^{\infty} f(0)\delta(x)g(x)\,\mathrm{d}x = f(0)\int_{-\infty}^{\infty} \delta(x)g(x)\,\mathrm{d}x = f(0)g(0)$$

Both sides are equal so the relationship is verified.

Derivatives Of Generalised Functions

Obviously there is no interpretation as rates of change, so given the close relationship with integrals, the derivatives of generalised functions are defined so that the method of integration by parts holds. For any good function g and any generalised function G,

$$\begin{aligned}\int_{-\infty}^{\infty} G'(x)g(x)\,\mathrm{d}x &= [G(x)g(x)]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} G(x)g'(x)\,\mathrm{d}x \\ &= -\int_{-\infty}^{\infty} G(x)g'(x)\,\mathrm{d}x\end{aligned}$$

Applying this to the generalised functions defined above,

- For the Heaviside function

$$\begin{aligned}\int_{-\infty}^{\infty} H'(x)g(x)\,\mathrm{d}x &= -\int_{-\infty}^{\infty} H(x)g'(x)\,\mathrm{d}x \\ &= -\int_{0}^{\infty} g'(x)\,\mathrm{d}x \\ &= -\,[g(x)]_0^{\infty} = g(0) \\ &= \int_{-\infty}^{\infty} \delta(x)g(x)\,\mathrm{d}x\end{aligned}$$

  So the derivative of the Heaviside function is the Dirac delta function.

- For the Dirac delta function

$$\begin{aligned}\int_{-\infty}^{\infty} \delta'(x)g(x)\,\mathrm{d}x &= -\int_{-\infty}^{\infty} \delta x g' x\,\mathrm{d}x \\ &= -g'(0) \\ &= \int_{-\infty}^{\infty} -\delta(x)\end{aligned}$$

  This shows that the Dirac delta function has no derivative in terms of generalised functions.

- Defining the modulus function $|x| = x \operatorname{sgn} x$ its derivative can be found as follows.

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}|x| &= \frac{\mathrm{d}}{\mathrm{d}x}(x \operatorname{sgn} x) \\
&= x\frac{\mathrm{d}}{\mathrm{d}x}(\operatorname{sgn} x) + \operatorname{sgn} x \quad \text{from the product rule} \\
&= x\frac{\mathrm{d}}{\mathrm{d}x}\left(2H(x) - I(x)\right) + \operatorname{sgn} x \\
&= 2x\delta(x) - \operatorname{sgn} x \\
&= \operatorname{sgn} x \quad \text{from (56) with } f(x) = x
\end{aligned}$$

(14.4.3) Fourier Transforms

Transform & Inverse Transform

Definition 57 *For a good function f, the Fourier transform is given by*

$$\mathcal{F}\left(f(x)\right) = \int_{-\infty}^{\infty} e^{ikx} f(x)\,\mathrm{d}x = \lim_{n\to\infty}\int_{-n}^{n} e^{ikx} f(x)\,\mathrm{d}x$$

which is a function of k.

From the definition it is possible to calculate Fourier transforms directly. For example,

$$\begin{aligned}
\mathcal{F}(1) &= \int_{-\infty}^{\infty} e^{ikx}\,\mathrm{d}x \\
&= \lim_{n\to\infty}\left[\frac{e^{ikx}}{ik}\right]_{-n}^{n} \\
&= \lim_{n\to\infty}\left(\frac{e^{ikn} - e^{-ink}}{ik}\right) \\
&= \lim_{n\to\infty}\left(\frac{2\sin(kn)}{k}\right) \\
&= 2\pi\lim_{n\to\infty}\left(\frac{\sin nk}{k\pi}\right) = 2\pi\delta(k)
\end{aligned}$$

$$\begin{aligned}
\mathcal{F}\left(e^{-|x|}\right) &= \int_{-\infty}^{0} e^{ikx}e^{x} + \int_{0}^{\infty} e^{ikx}e^{-x} \\
&= \left[\frac{e^{x(ik+1)}}{ik+1}\right]_{-\infty}^{0} + \left[\frac{e^{x(ik-1)}}{ik-1}\right]_{0}^{\infty} \\
&= \frac{1}{ik+1} - \frac{1}{ik-1} \\
&= \frac{2}{1+k^2}
\end{aligned}$$

It is obvious from the definition that the Fourier transform is a linear operator, and note now the following useful result. Suppose that $\mathcal{F}\left(f(x)\right) = \hat{f}(k)$, then,

$$\mathcal{F}\left(f(ax)\right) = \int_{-\infty}^{\infty} e^{ikx} f(ax)\,\mathrm{d}x$$

say $y = ax$ so $\dfrac{\mathrm{d}y}{\mathrm{d}x} = a$ and hence

$$\begin{aligned}
&= \int_{-\infty}^{\infty} e^{ik\frac{y}{a}} f y \frac{1}{a}\,\mathrm{d}y \\
&= \frac{1}{a}\int_{-\infty}^{\infty} e^{iy\frac{k}{a}} f(y)\,\mathrm{d}y \\
&= \frac{1}{a}\hat{f}\left(\frac{k}{a}\right)
\end{aligned}$$

More importantly still is the inverse transform.

$$\begin{aligned}\int_{-\infty}^{\infty} e^{-ikx}\mathcal{F}(f)\ \mathrm{d}k &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-ikx}e^{ikt}f(x)\ \mathrm{d}t\ \mathrm{d}k \\ &= \int_{-\infty}^{\infty} f(t)\int_{-\infty}^{\infty} e^{ik(t-x)}\ \mathrm{d}k\ \mathrm{d}t\end{aligned}$$

Now, the inner integral is in the form $\int_{-\infty}^{\infty} e^{ikz}\ \mathrm{d}z = 2\pi\delta(k)$, so by considering $z = t - x$,

$$= \int_{-\infty}^{\infty} f(t)2\pi\delta(t-x)\ \mathrm{d}t$$

Considering again $z = t - x$,

$$\begin{aligned}&= 2\pi\int_{-\infty}^{\infty} f(z+x)\delta(z)\ \mathrm{d}z \\ &= 2\pi f(x)\end{aligned}$$

From this calculation it is clear that

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ikx}\mathcal{F}(f)\ \mathrm{d}k$$

While this is clearly simpler than the inverse Laplace transform, it usually produces very difficult integrals.

Using Fourier Transforms

As with Laplace transforms, Fourier transforms can be used to solve differential equations. However, notice that the Laplace transform confines itself to $t > 0$—the range of integration on its defining integral. This is not so with Fourier transforms, and this removes certain constraints. To solve differential equations it is necessary to find the Fourier transform of a derivative, so using integration by parts,

$$\begin{aligned}\mathcal{F}\left(f'(x)\right) &= \int_{-\infty}^{\infty} e^{ikx}f'(x)\ \mathrm{d}x \\ &= \left[e^{ikx}f(x)\right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} ike^{ikx}f(x)\ \mathrm{d}x \\ &= -ik\mathcal{F}(f)\end{aligned}$$

Furthermore,

$$\mathcal{F}\left(f''\right) = -ik\mathcal{F}\left(f'(x)\right) = (-ik)(-ik)\mathcal{F}(f) = -k^2\mathcal{F}(f)$$

The convolution can also be defined, again with the advantage over the Laplace equivalent of being full range. Define

$$f * g = \int_{-\infty}^{\infty} f(y)g(x-y)\ \mathrm{d}y$$

now finding the Fourier transform,

$$\mathcal{F}(f*g) = \int_{x=-\infty}^{\infty} e^{ikx}\int_{y=-\infty}^{\infty} f(y)g(x-y)\ \mathrm{d}y\ \mathrm{d}x$$

Since the limits are independent, the integrals can be rearranged to give

$$\begin{aligned} &= \int_{-\infty}^{\infty} f(y) \int_{-\infty}^{\infty} e^{ikx} g(x-y)\, \mathrm{d}x\, \mathrm{d}y \qquad \text{now put } z = x - y, \\ &= \int_{-\infty}^{\infty} e^{iky} f(y)\, \mathrm{d}y \int_{-\infty}^{\infty} e^{ikz} g(z)\, \mathrm{d}z \\ &= \mathcal{F}(f)\, \mathcal{F}(g) \end{aligned}$$

Green's Function For Linear Operators

Fourier transforms can be used to work with some quite peculiar differential operators. For example

$$L \equiv \frac{\mathrm{d}^2}{\mathrm{d}x^2} - 1 \quad \text{so} \quad LG = \frac{\mathrm{d}^2 G}{\mathrm{d}x^2} - G$$

Definition 58 *A Green function G is a good function such that where L is a linear differential operator and δ is the Dirac delta function, $LG = \delta$.*

Taking Fourier transforms allows such an equation to be solved, with G finally being found by evaluating an inverse Fourier transform. The Fourier transform of the Dirac delta function is therefore of interest,

$$\mathcal{F}(\delta) = \int_{-\infty}^{\infty} e^{ikx} \delta(x)\, \mathrm{d}x = \left(e^{ikx} \right)\Big|_0 = 1$$

Theorem 59 *For functions ϕ and P, and linear differential operator L, the solution to the equation $L\phi = P$ is $\phi = G * P$.*

Proof. The proof is really a matter of showing that $L(G * P) = P$.

$$L(G * P) = (LG) * P = \delta * P = P$$

with the last step following because

$$\delta * P = \int_{-\infty}^{\infty} \delta(y) P(x - y)\, \mathrm{d}y = P(x)$$

□

The Dirichlet Problem For A Differential Equation

As discussed in Chapter **??** there are many physical applications of Laplace's equation, $\nabla^2 \phi = 0$. The Dirichlet problem for the half plane involves the solution of this with conditions $\phi = f(x)$ when $y = 0$ and $\phi \to 0$ as $y \to \infty$, working in the xy plane with $y \geqslant 0$.

Say $\phi = \phi(x, y)$ then taking the Fourier transform of ϕ for the variable x,

$$\mathcal{F}(\phi) = \hat{\phi}(k, y) = \int_{-\infty}^{\infty} e^{ikx} \phi(x, y)\, \mathrm{d}x$$

Hence taking the Fourier transform of the equation to be solved,

$$\begin{aligned}
\mathcal{F}\left(\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}\right) &= 0\\
\mathcal{F}\left(\frac{\partial^2\phi}{\partial x^2}\right)+\frac{\partial^2}{\partial y^2}\left(\mathcal{F}\left(\phi\right)\right) &= 0\\
-k^2\hat{\phi}+\frac{\partial^2\hat{\phi}}{\partial y^2} &= 0\\
\frac{\partial^2\hat{\phi}}{\partial y^2}-|k|^2\hat{\phi} &= 0
\end{aligned}$$

Clearly the solution to this is $\hat{\phi} = Ae^{|k|y} + Be^{-|k|y}$ where A and B are functions of k. Now, since $\phi \to 0$ as $y \to \infty$—a condition which is not effected by the Fourier transform since it does not involve x—it must be the case that $A(k) \equiv 0$.

The remaining equation $\hat{\phi} = B(k)e^{-|k|y}$ must now be made to satisfy the second condition, which becomes $\hat{\phi}(k,0) = \hat{f}(k)$, where $\mathcal{F}\left(f\right) = \hat{f}$. Clearly this gives $B = \hat{f}$, so $\hat{\phi}(k,y) = \hat{f}(k)e^{-|k|y}$.

The next step is to invert the transform, so using convolution,

$$\mathcal{F}\left(f * g\right) = \mathcal{F}\left(f\right)\mathcal{F}\left(g\right) \quad \text{gives} \quad f * g = \mathcal{F}^{-1}\left(\mathcal{F}\left(f\right)\mathcal{F}\left(g\right)\right)$$

In order to use this it is first necessary to find the inverse transforms of $\hat{f}$ (which is trivial), and of $e^{-|k|y}$, which is not so easy.

$$\begin{aligned}
\mathcal{F}^{-1}\left(e^{-|k|y}\right) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ikx}e^{-|k|y}\,\mathrm{d}k\\
&= \frac{1}{2\pi}\int_{-\infty}^{0} e^{k(y-ix)}\,\mathrm{d}x+\frac{1}{2\pi}\int_{0}^{\infty} e^{-k(y+ix)}\,\mathrm{d}x\\
&= \frac{1}{2\pi}\left[frace^{k(y-ix)}y-ix\right]_{-\infty}^{0}+\frac{1}{2\pi}\left[frace^{-k(y+ix)}-y-ix\right]_{0}^{\infty}\\
&= \frac{1}{2\pi}\frac{1}{y-ix}+\frac{1}{2\pi}\frac{-1}{-y-ix}\\
&= \frac{y}{\pi(y^2+x^2)}
\end{aligned}$$

Now returning to use convolution,

$$\begin{aligned}
\hat{\phi} &= \hat{f}e^{-|k|y}\\
\mathcal{F}\left(\phi\right) &= \mathcal{F}\left(f\right)\mathcal{F}\left(\frac{y}{\pi(y^2+x^2)}\right)\\
\phi &= f * \frac{y}{\pi\left(y^2+x^2\right)}\\
&= \int_{-\infty}^{\infty} f(\xi)\frac{y}{\pi\left(y^2+(x-\xi)^2\right)}\,\mathrm{d}\xi\\
&= \frac{y}{\pi}\int_{-\infty}^{\infty}\frac{f(\xi)}{y^2+(x-\xi)^2}\,\mathrm{d}\xi
\end{aligned}$$

Which is the solution to the problem.

The Neumann Problem For Differential Equations

The Neumann problem differers from the Dirichlet problem in the conditions given. Again the solution to $\nabla^2\phi = 0$ is sought but this time with condition $\frac{\partial\phi}{\partial y} = f(x)$ at $y = 0$.

To begin with, the solution takes much the same form as that of the Dirichlet problem, and again, $\hat{\phi}(k,y) = B(k)e^{-|k|y}$. Transforming the initial condition,

$$\begin{aligned} \mathcal{F}\left(\frac{\partial\phi}{\partial y}\right) &= \mathcal{F}(f) \quad \text{when } y = 0 \\ \frac{\partial\hat{\phi}}{\partial y} &= \hat{f}(k) \quad \text{when } y = 0 \\ \text{giving} \quad B(k)e^{-|k|0}(-|k|) &= \hat{f} \\ B(k) &= \frac{\hat{f}}{-|k|} \end{aligned}$$

To find the inverse Fourier transform of $\frac{\hat{f}}{-|k|}$ recall that

$$e^{-|k|y} = \frac{1}{\pi}\int_{-\infty}^{\infty} e^{ikx}\frac{y}{y^2+x^2}\,\mathrm{d}x$$

Now, the left hand side will become the expression to be inverse transformed if integrated with respect to y, hence

$$\begin{aligned} \int^y e^{-|k|y}\,\mathrm{d}y &= \frac{1}{\pi}\int^y\int_{-\infty}^{\infty} e^{ikx}\frac{y}{y^2+x^2}\,\mathrm{d}x\,\mathrm{d}y \\ \frac{e^{-|k|y}}{-|k|} &= \frac{1}{\pi}\int_{-\infty}^{\infty}\int^y e^{ikx}\frac{y}{y^2+x^2}\,\mathrm{d}y\,\mathrm{d}x \\ &= \frac{1}{\pi}\int_{-\infty}^{\infty} e^{ikx}\frac{1}{2}\ln\left(y^2+x^2\right)\,\mathrm{d}x \\ \frac{-e^{-|k|y}}{|k|} &= \frac{1}{2\pi}\mathcal{F}^{-1}\left(\ln\left(y^2+x^2\right)\right) \end{aligned}$$

Hence using convolution,

$$\phi(x,y) = \int_{-\infty}^{\infty} f(\xi)\frac{1}{2\pi}\ln\left(y^2+(x-\xi)^2\right)\,\mathrm{d}\xi$$

(14.5) Calculus Of Variations

(14.5.1) Definition & Uses

Take for example the question “what is the shortest distance between two points in the xy plane?” In the most general way this question can be solved by minimising the value of the integral

$$\int_{x_1}^{x_2}\sqrt{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x$$

which finds the length of the function $y = f(x)$ which joins x_1 to x_2.

The problem addressed in this section is to find the function y such that

$$I(y) = \int_a^b f(x,y,y')\,\mathrm{d}x$$

has a stationary point. Note that $I\colon [a,b] \to \mathbb{R}$ is called a functional.

Lemma 60 *If g is continuous on $[x_1, x_2]$ and $\int_{x_1}^{x_2} \eta(x)g(x)\,\mathrm{d}x = 0$ for all differentiable functions η with the property $\eta(x_1) = \eta(x_2) = 0$ then $g(x) \equiv 0$ on $[x_1, x_2]$.*

Theorem 61 *For a functional $I(y) = \int_a^b f(x, y, y')\,\mathrm{d}x$, the function $y(x)$ which gives a stationary value is determined by the differential equation*

$$\frac{\partial f}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial f}{\partial y'}\right) = 0 \quad \forall x \in (a,b)$$

This is called the Euler-Lagrange equation.

Proof. Suppose that the function y makes the functional I have a stationary point. Consider a perturbation about this point, $y + \varepsilon\eta$ where η is some other function of x and ε is a parameter.

$$\begin{aligned} \text{say} \quad I(y) &= \int_a^b f(x, y, y')\,\mathrm{d}x \\ \text{and define} \quad J(\varepsilon) = I(y+\varepsilon\eta) &= \int_a^b f(x, y+\varepsilon\eta, y'+\varepsilon\eta')\,\mathrm{d}x \end{aligned}$$

It is required that $\eta(a) = \eta(b) = 0$. At a stationary value, $J'(0) = 0$, so differentiating with respect to ε,

$$\begin{aligned} J'(\varepsilon) &= \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\int_a^b f(x, y+\varepsilon\eta, y'+\varepsilon\eta')\,\mathrm{d}x \\ &= \int_a^b \frac{\partial f}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}\varepsilon} + \frac{\partial f}{\partial(y+\varepsilon\eta)}\frac{\mathrm{d}}{\mathrm{d}\varepsilon}(y+\varepsilon\eta) + \frac{\partial f}{\partial(y'+\varepsilon\eta')}\frac{\mathrm{d}}{\mathrm{d}\varepsilon}(y'+\varepsilon\eta')\,\mathrm{d}x \\ &= \int_a^b \frac{\partial f}{\partial(y+\varepsilon\eta)}\eta + \frac{\partial f}{\partial(y'+\varepsilon\eta')}\eta'\,\mathrm{d}x \\ J'(0) &= \int_a^b \frac{\partial f}{\partial y}\eta + \frac{\partial f}{\partial y'}\eta'\,\mathrm{d}x \\ \text{but} \quad \int_a^b \frac{\partial f}{\partial y'}\eta'\,\mathrm{d}x &= \left[\frac{\partial f}{\partial y'}\eta\right]_a^b - \int_a^b \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial f}{\partial y'}\right)\,\mathrm{d}x \\ \text{hence} \quad J'(0) &= \int_a^b \left(\frac{\partial f}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial f}{\partial y'}\right)\right)\eta\,\mathrm{d}x \end{aligned}$$

Since η is any function, $J'(0) = 0$ only when

$$\frac{\partial f}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial f}{\partial y'}\right) = 0$$

Hence the result. □

Note that the partial derivatives are partial—the chain rule does not need to be used. It is possible that x does not appear explicitly in f, giving $f(y, y')$. In this case differentiation with respect to x must be done by the chain rule, so for example

$$\begin{aligned} \frac{\mathrm{d}f}{\mathrm{d}x} &= \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' \\ &= y'\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial f}{\partial y'}\right) + \frac{\partial f}{\partial y'}y'' \quad \text{from the Euler-Lagrange equation} \\ &= \frac{\mathrm{d}}{\mathrm{d}x}\left(y'\frac{\partial f}{\partial y'}\right) \quad \text{by reversing the product rule} \\ \text{integrating, } f &= y'\frac{\partial f}{\partial y'} + k \end{aligned}$$

This is called the first integral of the Euler-Lagrange equation.

Having found a stationary point it has to be determined whether it is a maximum or a minimum. For a minimum,

$$\frac{\partial^2 f}{\partial y^2} > 0 \quad \text{and} \quad \frac{\partial^2 f}{\partial y^2}\frac{\partial^2 f}{\partial (y')^2} - \left(\frac{\partial^2 f}{\partial y \partial y'}\right)^2 > 0$$

or alternatively,

$$\frac{\partial^2 f}{\partial (y')^2} > 0 \quad \text{and} \quad \frac{\partial^2 f}{\partial y^2}\frac{\partial^2 f}{\partial (y')^2} - \left(\frac{\partial^2 f}{\partial y \partial y'}\right)^2 > 0$$

For a functional with integrand $f(x, y, y', y'')$ the Euler-Lagrange equation becomes

$$\frac{\partial f}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial f}{\partial (y')}\right) + \frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(\frac{\partial f}{\partial (y'')}\right) = 0$$

Isoperimetric Problems

As well as minimising a functional, it may also be necessary to obey a constraint of the form

$$\int_a^b y \,\mathrm{d}x = k$$

In this case define

$$\hat{I} = \int_a^b f(x, y, y') \,\mathrm{d}x + \lambda\left(\int_a^b y \,\mathrm{d}x - k\right) = \int_a^b f + \lambda y - \lambda k \,\mathrm{d}x$$

where the parameter λ is a Lagrange multiplier. The new functional $\hat{I}$ can now be worked on. The resulting function y will also be a function of λ, but the condition $\int_a^b y \,\mathrm{d}x = k$ can now be used to find a value for λ.