Chapter 17

MSMXG4 Complex Variable Theory

(17.1) Complex Functions

(17.1.1) Complex Numbers

First of all it is convenient to review some of the basis properties of the complex numbers. A complex number is an ordered pair of real numbers, written $x + iy$ where $x, y \in \mathbb{R}$ and $i = \sqrt{-1}$. The set of complex numbers is denoted as $\mathbb{C}$. The usual properties of a field hold for the complex numbers, i behaves as would be expected for an algebraic factor. When performing calculations it is of course useful to remember that $i^2 = -1$.

As an ordered pair, a complex number can have a graphical representation in a plane. Where $z \in \mathbb{C}$ such that $z = x + iy$ the Cartesian plane can be modified so that the x axis is the real axis and the y axis is the imaginary axis. This is called the Argand diagram.

Definition 1 *If $zx + iy \in \mathbb{C}$ then the absolute value (or modulus) of z, written $\|z\|$, is the real number $\sqrt{x^2 + y^2}$.*

Definition 2 *If $zx + iy \in \mathbb{C}$ then $\bar{z} = x - iy$ is the complex conjugate of z.*

From the definition of a complex conjugate the following can readily be shown.

Assertion 3

1. $\bar{z}_1 + \bar{z}_2 = \overline{z_1 + z_2}$
2. $z\bar{z} = |z|^2$
3. $|z_1 z_2| = |z_1|\,|z_2|$
4. $\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$
5. $\left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|}$
6. $|z_1 + z_2| \leqslant |z_1| + |z_2|$
7. $Re(z) = \dfrac{z + \bar{z}}{2}$ *and similarly* $Im(z) = \dfrac{z - \bar{z}}{2}$

Having represented complex numbers in the Cartesian plane, a logical next step is to express them in a polar form. This is done in the obvious way giving $x + iy = r(\cos\theta + i\sin\theta)$ where $\theta = \tan^{-1}\dfrac{y}{x}$ and $r = \|z\|$. However, θ is not unique, as it can be readily replaced by $\theta + 2k\pi$ where $k \in \mathbb{Z}$. The *set* of all possible values of θ is called $\arg(z)$, and the following definition is therefore made.

Definition 4 *The unique element of* $\arg(z)$ *which lies in the range* $(-\pi, \pi]$ *is called the* principal argument *of z, and is denoted as* $\operatorname{Arg}(z)$.

The following identity should be noted.

Assertion 5 $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2) = \{\theta_1 + \theta_2 \mid \theta_1 \in \arg(z_1), \theta_2 \in \arg(z_2)\}$

However, there is another form which makes calculations even easier.

Definition 6 (Exponential Form Of A Complex Number) $e^{i\theta} = \cos\theta + i\sin\theta$

It can be shown that $\left(e^{i\theta_1}\right)\left(e^{i\theta_2}\right) = e^{i(\theta_1+\theta_2)}$, but this must be done from the formulae for $\sin(a+b)$ and $\cos(a+b)$ in order to avoid a circular argument.

Theorem 7 (DeMoivre's Theorem)

$$(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)$$

Proof. The easy way is to observe that $\left(e^{i\theta}\right)^n = e^{in\theta}$. Alternatively, use proof by induction. □

Sufficient theory is now available to solve the following kind of problem.

Example 8 *Find the 3rd roots of the complex number* $1-i$.

Proof. Solution Seek z such that $z^3 = 1-i$. Now, $|1-i| = \sqrt{1^2+(-1)^2} = \sqrt{2}$ and $\arg(1-i) = \tan^{-1}\left(\frac{1}{-1}\right) = \frac{-\pi}{4}$, being careful to make sure that the value of θ is in the correct quadrant. Therefore,

$$z^3 = \sqrt{2}\exp\left(i(\frac{-\pi}{4} + 2k\pi)\right) \quad \text{where } k \in \mathbb{Z}$$
$$z = \sqrt[6]{2}\exp\left(\frac{i(\frac{-\pi}{4} + 2k_1\pi)}{3} + 2k_2\pi\right) \quad k_1, k_2 \in \mathbb{Z}$$

So the principal arguments for the roots are $\frac{-\pi}{12}, \frac{-9\pi}{12}, \frac{7\pi}{12}$ giving as the answers

$$z_1 = \sqrt[6]{2}e^{\frac{-\pi}{12}} \qquad z_2 = \sqrt[6]{2}e^{\frac{-3\pi}{4}} \qquad z_3 = \sqrt[6]{2}e^{\frac{7\pi}{12}}$$ □

(17.1.2) Complex Valued Functions Of A Complex Variable

Having gained experience with complex numbers, it is natural to extend to functions of these numbers. As usual a function $f : S \to T$ has the properties

- $\forall s \in S \quad \exists! t \in T$ which is the image of s under the function.
- S is called the domain and T is called the range.

It is evident that any complex function must have an expression of the form

$$f(x+iy) = u(x,y) + iv(x,y)$$
$$\text{where } u : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$$
$$\text{and } v : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$$

Perhaps the most basic type of function is the polynomial, which has an interpretation in the complex sense,

Definition 9 *Suppose that* $n \geqslant 0$ *and* $a_0, a_1, \ldots, a_n \in \mathbb{C}$ *where* $a_n \neq 0$. *Then*

$$P(z) = a_0 + a_1 z + \cdots + a_n z^n$$

is a polynomial of degree n.

Furthermore, if $R(z)$ and $S(z)$ are polynomials, then $T(z) = \dfrac{R(z)}{S(z)}$ is a rational function, wherever it is defined.

Drawing a graph of a complex function is difficult, as it is a map from a plane to a plane. Translations and rotations are easy to visualise, but for anything more complicated drawing a graph is near impossible. Despite this handicap, it is possible to produce the usual results of analysis.

Definition 10 *Let z_0 be a complex number, and $\varepsilon > 0$ be a real number. The ε-neighbourhood of z_0 is defined by the set*

$$B(z_0, \varepsilon) = \{z \mid |z - z_0| < \varepsilon\}$$

So a complex number z is said to be in the neighbourhood of z_0 if $\exists \varepsilon > 0$ such that $z \in B(z_0, \varepsilon)$.

Definition 11 *Let $S, T \subseteq \mathbb{C}$, let f be a function $f : S \to T$, and suppose that $z_0 \in \mathbb{C}$.*

$$\begin{aligned} & \lim_{z \to z_0} f(z) = w_0 \quad (w_0 \in T) \\ \Leftrightarrow \quad & \textit{given} \quad \varepsilon > 0 \quad \exists \delta > 0 \quad \textit{such that} \quad |f(z) - w_0| < \varepsilon \textit{ whenever} \quad 0 < |z - z_0| < \delta \\ \Leftrightarrow \quad & f(z) \in B(w_0, \varepsilon) \quad \textit{whenever} \quad z \in (B(z_0, \delta) \setminus \{z_0\}) \cap S \end{aligned}$$

The usual results for limits can now be proved.

Theorem 12 *Suppose that f is a function, and $\lim_{z \to z_0} f(z)$ exists, then it is unique.*

Proof. Suppose the result is false, so that say

$$\lim_{z \to z_0} f(z) = w_1 \quad \text{and} \quad \lim_{z \to z_0} f(z) = w_2$$

where $w_1 \neq w_2$. The situation is something like that shown in Figure 1.
By the definition of a limit,

$$\forall \varepsilon > 0 \ \exists \delta_1 > 0 \text{ such that } 0 < |z - z_0| < \delta_1 \ \Rightarrow \ |f(z) - w_1| < \varepsilon$$

And similarly,

$$\forall \varepsilon > 0 \ \exists \delta_2 > 0 \text{ such that } 0 < |z - z_0| < \delta_2 \ \Rightarrow \ |f(z) - w_2| < \varepsilon$$

Let $\gamma = |w_1 - w_2|$ and consider $\varepsilon < \dfrac{\gamma}{2}$. Clearly

$$\begin{aligned} |f(z) - w_1| &< \frac{\gamma}{2} \\ |f(z) - w_2| &< \frac{\gamma}{2} \\ \text{adding, } |f(z) - w_1| + |f(z) - w_2| &< \gamma \end{aligned} \tag{13}$$

However,

$$\begin{aligned} |w_1 - w_2| &= |w_1 - f(z) + f(z) - w_2| \\ &\leqslant |f(z) - w_1| + |f(z) - w_2| \quad \text{by the triangle inequality} \end{aligned}$$

From equation (13) this gives

$$\gamma < |f(z) - w_1| + |f(z) - w_2| < \gamma$$

This is clearly a contradiction, hence the theorem holds. $\square$

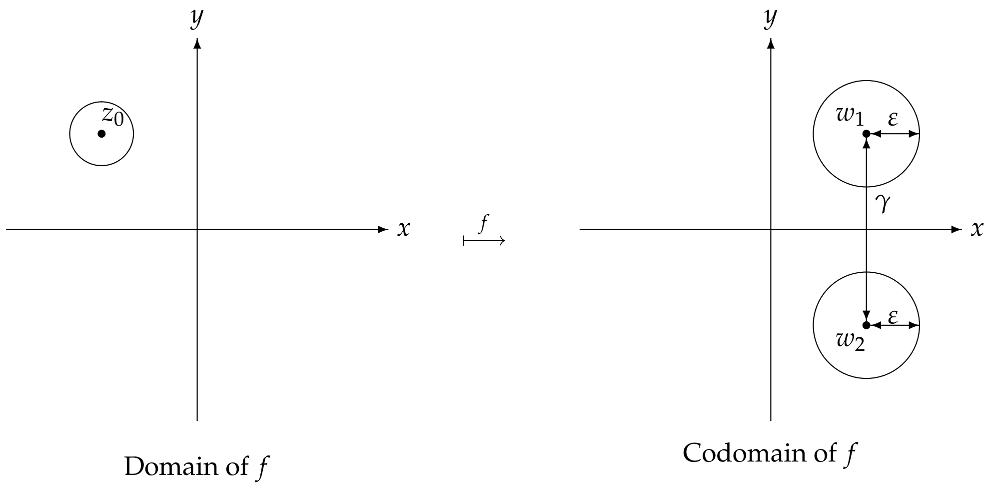

Figure 1: The limit of a function must be unique

Theorem 14 *If $z = x + iy$ and $f(z) = u(x, y) + iv(x, y)$, then where $z_0 = x_0 + iy_0$ and $w_0 = u_0 + iv_0$,*

$$\lim_{z\to z_0} = w_0 \quad \Leftrightarrow \quad \lim_{(x,y)\to(x_0,y_0)} u(x, y) = u_0 \quad \textit{and} \quad \lim_{(x,y)\to(x_0,y_0)} v(x, y) = v_0$$

The proof of this theorem is omitted, but it follows much the same form as an example of finding a limit.

Theorem 15 (The Algebra Of Limits) *Suppose that f and g are functions with $\lim_{z\to z_0} f(z) = \phi$ and $\lim_{z\to z_0} g(z) = \rho$. Then,*

1. $\lim_{z\to z_0} (f + g)(z) = \phi + \rho$
2. $\lim_{z\to z_0} kf(z) = k\phi \quad k \in \mathbb{C}$
3. $\lim_{z\to z_0} f(z)g(z) = \phi\rho$
4. $\lim_{z\to z_0} \frac{f}{g}(z) = \frac{\phi}{\rho} \quad \rho \neq 0$

Proof.

1. By the definition of a limit,

$$\forall\varepsilon > 0 \quad \exists\delta_1 > 0 \quad \text{such that} \quad |f(z) - \phi| < \varepsilon \quad \text{whenever} \quad 0 < |z - z_0| < \delta_1$$

and similarly,

$$\forall\varepsilon > 0 \quad \exists\delta_2 > 0 \quad \text{such that} \quad |g(z) - \rho| < \varepsilon \quad \text{whenever} \quad 0 < |z - z_0| < \delta_2$$

Hence taking $\delta \leqslant \min(\delta_1, \delta_2)$,

$$\begin{aligned} |f(z) - \phi| < \frac{\varepsilon}{2} \quad \text{and} \quad |g(z) - \rho| &< \frac{\varepsilon}{2} \quad \text{for} \quad \delta \leqslant \min(\delta_1, \delta_2) \\ |f(z) - \phi| + |g(z) - \rho| &< \varepsilon \\ |f(z) + g(z) - (\phi + \rho)| &< \varepsilon \quad \text{by the triangle inequality} \end{aligned}$$

and hence by the definition of a limit, the result holds.

2. Again from the definition of a limit,

$$
\begin{aligned}
|f(z) - \phi| &< \frac{\varepsilon}{|k|} \quad \text{for some } \delta > 0 \text{ where} \quad 0 < |z - z_0| < \delta \\
|k|\,|f(z) - \phi| &< \varepsilon \\
|kf(z) - k\phi| &< \varepsilon
\end{aligned}
$$

And so by the definition of a limit, the result holds.

3. From the definition of a limit,

$$
\begin{aligned}
|f(z) - \phi| &< \varepsilon \quad \text{where} \quad |z - z_0| < \delta_1 \\
|g(z) - \rho| &< \varepsilon \quad \text{where} \quad |z - z_0| < \delta_2 \\
|g(z) - \rho| + |\rho| &< \varepsilon + |\rho| \\
|g(z)| &< \varepsilon + |\rho| \quad \text{by the triangle inequality} \\
\text{now,} \quad |f(z)g(z) - \phi\rho| &= |f(z)g(z) - \phi g(z) + \phi g(z) - \phi\rho| \\
&\leqslant |\phi|\,|g(z) - \rho| + |g(z)|\,|f(z) - \phi| \\
&< |\phi|\varepsilon + \varepsilon\,(\varepsilon + |\rho|) \\
&< \varepsilon_1
\end{aligned}
$$

since ε_1 can be made arbitrarily small. Hence by the definition of a limit, the result holds.

4. As above,

$$
\begin{aligned}
|f(z) - \phi| &< \varepsilon \quad \text{where} \quad |z - z_0| < \delta_1 \\
|f(z)| &< \varepsilon + |\phi| \\
|g(z) - \rho| &< \varepsilon \quad \text{where} \quad |z - z_0| < \delta_2 \\
|g(z)| &< \varepsilon + |\rho|
\end{aligned}
$$

now,

$$
\begin{aligned}
\left|\frac{f(z)}{g(z)} - \frac{\phi}{\rho}\right| &= \left|\frac{f(z)}{g(z)} - \frac{f(z)}{\rho} + \frac{f(z)}{\rho} - \frac{\phi}{\rho}\right| \\
&= \left|f(z)\left(\frac{1}{g(z)} - \frac{1}{\rho}\right) + \frac{1}{\rho}\left(f(x) - \phi\right)\right| \\
&< \frac{|f(z)|}{|g(z)||\rho|}\varepsilon + \frac{1}{|\rho|} \\
epsilon \\
&< \varepsilon_1
\end{aligned}
$$

since ε_1 can be made arbitrarily small. Hence by the definition of a limit, the result holds. □

An instant application of this theorem is to polynomial functions, which are composed of products and sums. It also follows that if $\lim_{z \to z_0} f(z) = w_0$ then $\lim_{z \to z_0} |f(z)| = |w_0|$, which is shown using the triangle inequality.

(17.1.3) Infinity

Defining limits for finite points in the complex plane is all very well, however, in a plane 'infinity' is not really defined. On the real line this is not a problem, the real line has order but a plane does not — changing from a line to a plane introduces such problems.

An obvious way to re-define 'infinity' is to say that the modulus of the complex number increases without bound. This idea is formulated rigorously in terms of the Riemann sphere.

Imagine the complex plane, and a sphere is 'pushed through' it at the origin, so that the central part of the plane is deformed into a hemisphere — all these points already lie on the sphere. Above the surface of the complex plane now protrudes a hemisphere, and any point in the remaining (flat) plane is now joined by a straight line to the top of the sphere, and so this line will intersect the sphere at some point. As this point of intersection approaches the top of the sphere, the complex number approaches 'infinity'.

Note that distances between complex numbers in the complex plane are not preserved for their corresponding points on the Riemann sphere. Since $\mathbb{C}$ does not contain any kind of 'infinity' — as indeed $\mathbb{R}$ does not — the extended complex numbers are defined to be $\mathbb{C} \cup \{\infty\}$. As would be expected, the following rules hold for 'infinity'.

$$a + \infty = \infty \qquad a - \infty = \infty \qquad a.\infty = \infty \qquad \frac{1}{0} = \infty \qquad \frac{a}{\infty} = 0$$

For any non-zero complex number a.

(17.2) Differential Calculus Of Complex Functions

(17.2.1) Continuity

A discussion of limits is clearly aimed towards differentiation, but before this the (obvious) definition of continuity is made.

Definition 16 *A function $f(z)$ is continuous at a point $z_0 \in \mathbb{C}$ if*

- $f(z_0)$ *is defined, and*
- $\lim_{z \to z_0} f(z) = f(z_0)$

A function is said to be continuous on a set $S \subseteq \mathbb{C}$ if it continuous at every point in S. Note by the algebra of limits it is easily seen that polynomial functions are continuous.

In the same way that a real function is defined on a subset of $\mathbb{R}$, a complex function may be defined on a subset of $\mathbb{C}$. However, the two dimensional nature of $\mathbb{C}$ makes the type of subset rather more complicated than the simple open, closed, or half-open intervals of $\mathbb{R}$.

Definition 17 *Suppose that $S \subseteq \mathbb{C}$ and $z \in S$.*

1. *Then one of the following holds,*
   (a) *there exists a neighbourhood of z which lies completely within S. In this case z is said to be in the interior of S.*
   (b) *there exists a neighbourhood of z which contains no points in S. In this case z is said to be in the exterior of S.*

   (c) *neither of the above two conditions hold, so that any neighbourhood of z contains points in S and points not in S. Such a z is said to be a boundary point of S.*

2. *The set of boundary points of S is called the boundary of S.*
3. *A subset S of $\mathbb{C}$ is open* $\Leftrightarrow \quad \forall z \in S \quad \exists \delta > 0 \quad \textit{such that} \quad B(z, \delta) \subseteq S$
4. *S is open if every point in S is an interior point.*
5. *S is closed if it contains all its boundary points.*
6. *S is connected if any two points in S can be joined by a polygonal path lying wholly in S.*
7. *S is a domain if it is both open and connected.*
8. *S is simply connected if it contains no holes*[*]
9. *a domain together with some (or all) of its boundary points is called a region.*

It is now possible to make a more useful definition regarding the continuity of a function

Definition 18 *A function defined on a region R is said to be continuous on R, provided that it is continuous at every point in R.*

The ideas of these different kinds of subsets of $\mathbb{C}$ makes meaningful the idea of bounding.

Definition 19 *A region R is bounded if $\exists r > 0$ such that $R \subset B(0, r)$.*

And similarly for a function,

Definition 20 *If a function f is defined on a region R, then f is bounded provided that*

$$\exists M > 0 \quad \forall z \in R \quad \textit{such that} \quad |f(z)| \leqslant M \quad \forall z \in R$$

It is clearly seen that where $f(x+iy) = u(x,y) + iv(x,y)$ is defined and is continuous on the closed bounded region R, then

$$|f(z)| = \sqrt{(u(x,y))^2 + (v(x,y))^2}$$

is continuous and attains its bounds.

(17.2.2) Derivatives

In the usual way,

Definition 21 *Let f be a function whose domain of definition contains the point z_0. The derivative of f at z_0 is defined by the limit*

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

If the limit exists, then its value is called the derivative of f at z_0 and f is said to be differentiable at z_0. Otherwise f is not differentiable at z_0.

As with real functions, if a complex function is differentiable then it is continuous. For the real case this is proved in Chapter **??** Theorem **??**.

[*]This is a rather poor definition, although it is good enough for present purposes. A more rigorous definition considers contracting cycles from any point in S.

Theorem 22 *Suppose that $f(x+iy) = u(x,y) + iv(x,y)$ is differentiable at the point $z_0 = x_0 + iy_0$. Then*

$$f'(z_0) = \left.\frac{\partial u}{\partial x}\right|_{(x_0,y_0)} + i\left.\frac{\partial v}{\partial x}\right|_{(x_0,y_0)} = \left.\frac{\partial v}{\partial y}\right|_{(x_0,y_0)} - i\left.\frac{\partial u}{\partial y}\right|_{(x_0,y_0)}$$

This theorem reveals that in order for a complex function to be differentiable there must be a special relationship between its real and imaginary parts.

Proof. Because f is differentiable at z_0, the limit can be taken through real values and through imaginary values and the value of the limit must be the same i.e.

case 1: Approaching z_0 through real values,

$$\begin{aligned}
\lim_{\substack{\Delta z \to 0 \\ \Delta z \in \mathbb{R}}} \frac{f(z_0+\Delta z) - f(z_0)}{\Delta z} &= \lim_{\Delta x \to 0} \frac{f(z_0+\Delta x) - f(z_0)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{u(x_0+\Delta x, y_0) + iv(x_0+\Delta x, y) - u(x_0,y_0) - iv(x_0,y_0)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{u(x_0+\Delta x, y_0) - u(x_0,y_0)}{\Delta x} + \frac{iv(x_0+\Delta x, y_0) - iv(x_0,y_0)}{\Delta x} \\
&= \left.\frac{\partial u}{\partial x}\right|_{(x_0,y_0)} + i\left.\frac{\partial v}{\partial x}\right|_{(x_0,y_0)}
\end{aligned}$$

Hence the result.

case 2: Approaching through imaginary values,

$$\begin{aligned}
\lim_{\substack{\Delta z \to 0 \\ \Delta \operatorname{Re} z \equiv 0}} \frac{f(z_0+\Delta z) - f(z_0)}{\Delta z} &= \lim_{\Delta y \to 0} \frac{f(z_0+\Delta y) - f(z_0)}{i\Delta y} \\
&= \lim_{\Delta y \to 0} \frac{u(x_0, y_0+\Delta y) + iv(x_0, y+\Delta y) - u(x_0,y_0) - iv(x_0,y_0)}{i\Delta y} \\
&= \lim_{\Delta y \to 0} \frac{u(x_0, y_0+\Delta y) - u(x_0,y_0)}{i\Delta y} + \frac{iv(x_0+\Delta x, y_0) - iv(x_0,y_0)}{i\Delta y} \\
&= \lim_{\Delta y \to 0} \frac{-iu(x_0, y_0+\Delta y) + iu(x_0,y_0)}{\Delta y} + \frac{v(x_0+\Delta x, y_0) - v(x_0,y_0)}{\Delta y} \\
&= -i\left.\frac{\partial u}{\partial y}\right|_{(x_0,y_0)} + \left.\frac{\partial v}{\partial y}\right|_{(x_0,y_0)}
\end{aligned}$$

Hence the result. □

Definition 23 *If a complex function f is differentiable at a point z_0 and at every point in an open neighbourhood of z_0, then f is analytic at z_0.*

In the obvious way, f is said to be analytic on a domain D if it is analytic at every point of D. Furthermore, if f is analytic at all points of $\mathbb{C}$ then it is called entire.

If f is entire, then the partial derivatives calculated in Theorem 22 will exist everywhere. By equating real and imaginary parts,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \tag{24}$$

These are called the Cauchy-Riemann Equations. Conversely, if the Cauchy-Riemann equations hold at z_0 then f is differentiable at z_0 provided that the four partial derivatives are continuous, and u and v are continuous at z_0. Hence a function can be shown to be entire by showing that the Cauchy-Riemann equations hold for all $\mathbb{C}$.

Theorem 25 *Let f be an analytic function defined on a domain D. If $f'(z) = 0$ then the function is a constant.*

Proof. Since f is analytic the Cauchy-Riemann equations holds so,

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = -i\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

Hence equating real and imaginary parts

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = 0 \qquad \frac{\partial v}{\partial x} = \frac{\partial u}{\partial y} = 0$$

Now, $\dfrac{\partial u}{\partial x} = \dfrac{\partial u}{\partial y} = 0$ shows that u takes constant values on horizontal and vertical lines.

Furthermore, $\dfrac{\partial v}{\partial x} = \dfrac{\partial v}{\partial y} = 0$ shows that v takes constant values on horizontal and vertical lines.

Hence $f = u + v$ takes constant values on horizontal and vertical lines. Since the domain is connected, any two points can be joined with horizontal and vertical lines, and hence f is constant. □

For a fuller proof, it would be necessary to show that the function is constant on diagonal lines so that any two points of D can be joined with a polygonal line.

The algebra of limits shows the usual results for differentiation, and the usual rules for differentiation hold. In particular,

1. the derivative of a constant function is zero.
2. $\dfrac{\mathrm{d}}{\mathrm{d}z}(cf(z)) = c\dfrac{\mathrm{d}f}{\mathrm{d}z} \quad c \in \mathbb{C}$.
3. $\dfrac{\mathrm{d}}{\mathrm{d}z}(z^n) = nz^{n-1}$.
4. $\dfrac{\mathrm{d}}{\mathrm{d}z}(f(z) + g(z)) = \dfrac{\mathrm{d}f}{\mathrm{d}z} + \dfrac{\mathrm{d}g}{\mathrm{d}z}$.
5. $\dfrac{\mathrm{d}}{\mathrm{d}z}(f(z)g(z)) = f(z)\dfrac{\mathrm{d}g}{\mathrm{d}z} + g(z)\dfrac{\mathrm{d}f}{\mathrm{d}z}$ i.e. the product rule.
6. $\dfrac{\mathrm{d}}{\mathrm{d}z}\left(\dfrac{f(z)}{g(z)}\right) = \dfrac{g(z)\frac{\mathrm{d}f}{\mathrm{d}z} - f(z)\frac{\mathrm{d}g}{\mathrm{d}z}}{(f(z))^2}$ i.e. the quotient rule.
7. $\dfrac{\mathrm{d}}{\mathrm{d}z}(f \circ g(x)) = g'(f(z))f'(z)$. i.e. the chain rule.

(17.2.3) Harmonic Functions

Definition 26 *Let u be a real valued function of two real variables x and y. u is a Harmonic function if its first and second partial derivatives exist and are continuous, and obey the Laplace Equation,*

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

For an analytic function, from the Cauchy-Riemann equations,

$$\begin{aligned}
\frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y} \\
\text{so} \quad \frac{\partial^2 u}{\partial x^2} &= \frac{\partial^2 v}{\partial x \partial y} = \frac{\partial^2 v}{\partial y \partial x} \quad \text{by continuity} \\
&= \frac{\partial}{\partial y}\left(\frac{\partial v}{\partial x}\right) \\
&= \frac{\partial}{\partial y}\left(-\frac{\partial u}{\partial y}\right) \quad \text{by the other Cauchy-Riemann equation} \\
&= -\frac{\partial^2 u}{\partial y^2} \\
\text{hence} \quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} &= 0
\end{aligned}$$

So if f is an analytic complex function — so the Cauchy-Riemann equations hold — then its real and imaginary parts are harmonic functions.

Definition 27 *If two harmonic functions u and v obey the Cauchy-Riemann equations in a domain D, then where $f(x + iy) = u(x, y) + iv(x, y)$, v is the harmonic conjugate of u.*

From the Cauchy-Riemann equations it is evident that if v is a harmonic conjugate of u, then $-u$ is a harmonic conjugate of v.

(17.2.4) Transcendental Functions

The exponential function is defined as $e^z = e^{x+iy}$ and notice that this is equal to $e^x(\cos y + i \sin y)$.

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}z}(e^z) &= \frac{\partial}{\partial x} e^x \cos y + i \frac{\partial}{\partial x} e^x \sin y \quad \text{by Theorem 22} \\
&= e^x \cos y + i e^x \sin y \\
&= e^{x+iy} = e^z
\end{aligned}$$

So as would be expected, the derivative of e^z is e^z.

Definition 28 *The sine and cosine of a complex variable are defined as*

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i} \qquad \cos z = \frac{e^{iz} + e^{-iz}}{2}$$

From these definitions expressions for the tangent, cotangent, secant, and cosecant can be found.

The logarithm is the inverse function of the exponential, but in the complex case this is problematic since e^z is not a bijection — $e^z = e^{z+2k\pi i}$ where $k \in \mathbb{Z}$.

Definition 29 *The logarithm of a complex variable z is defined as*

$$\ln z = \ln|z| + i \arg z$$

which is multivalued and so is not a function. The principal logarithm is defined as

$$\operatorname{Ln} z = \ln|z| + i \operatorname{Arg} z$$

which is a function.

Notice that $e^{\operatorname{Ln} z} = e^{\ln|z|+i\operatorname{Arg} z} = |z|e^{i\operatorname{Arg} z} = z$ as would be expected. Since $\ln x$ is continuous for $x > 0$ and $\operatorname{Arg} z$ is continuous for $z \notin \mathbb{R}^-$, it follows that the complex logarithm is continuous on $\mathbb{C} \setminus \{\mathbb{R}_0^-\}$.

Theorem 30 *Let $U = \mathbb{C} \setminus \{\mathbb{R}_0^-\}$ then $\operatorname{Ln} z$ is analytic on U with derivative $\dfrac{1}{z}$.*

Proof. From the definition of a derivative,

$$\begin{aligned}
\lim_{z\to z_0} \frac{\operatorname{Ln} z - \operatorname{Ln} z_0}{z - z_0} &= \lim_{z\to z_0} \frac{\operatorname{Ln} z - \operatorname{Ln} z_0}{e^{\operatorname{Ln} z} - e^{\operatorname{Ln} z_0}} \\
&= \lim_{w\to w_0} \frac{w - w_0}{e^w - e^{w_0}}
\end{aligned}$$

from putting $w = \operatorname{Ln} z$ and $w_0 = \operatorname{Ln} z_0$. The subscript on the limit can be changed because the complex logarithm is continuous i.e. $\operatorname{Ln} z \to \operatorname{Ln} z_0$ and hence $w \to w_0$ as $z \to z_0$. The continuity of the exponential is also required.

$$\begin{aligned}
&= \lim_{w\to w_0} \left(\frac{e^w - e^{w_0}}{w - w_0}\right)^{-1} \\
&= \left(e^{w_0}\right)^{-1} \quad \text{because } e^z \text{ is differentiable with derivative } e^z. \\
&= \frac{1}{z_0}
\end{aligned}$$

which holds for any $z_0 \in U$. $\square$

It has been seen that the complex logarithm is not a proper function, and the principal logarithm was defined in the obvious way. However, there are many more ways to make the complex logarithm into a function.

Definition 31 *Suppose that U is a domain, and $f\colon U \to \mathbb{C}$ is a continuous function with the property $e^{f(z)} = z \quad \forall z \in U$. Then f is called a branch of the logarithm in U.*

Many possibilities for f are realised by making an appropriate restriction of the set $\arg z$. Define

$$\arg_\alpha z = \{\arg z \mid \arg z \in (\alpha - 2\pi, \alpha]\}$$

so that

$$\ln_\alpha z = \ln|z| + i\arg_\alpha z$$

Clearly this is a branch of the logarithm, but care must be taken in specifying on what domain. The set $\arg_\alpha z$ behaves rather like $\operatorname{Arg} z$, where it necessary to exclude the negative real axis, $\mathbb{R}^-$ — recall that $\operatorname{Arg} z \in (-\pi, \pi]$. In the case of $\arg_\alpha z$ the acceptable range of values starts and ends at the half line $\theta = \alpha$. Hence the domain U is the complex plane less this half line.

Logarithms are closely linked with expressions of the form a^z, in this case for some $a \in \mathbb{C}$. The exponent is defined as $a^z = e^{z\ln a}$ and the principal exponent by $a_p^z = e^{z\operatorname{Ln} a}$.

(17.3) Integral Calculus Of Complex Functions

(17.3.1) Paths & Contours

A real integral is usually interpreted as the area under a line, or the volume under a surface — see Chapters **??** and **??**. However, in the complex case no such interpretation is readily available.

Definition 32 *Where $[a, b] \subset \mathbb{R}$ is an interval on the real line, a function $\gamma\colon [a, b] \to \mathbb{C}$ is a path.*

A path may commonly be identified by its image rather than the function, which is the set $\mathit{\Gamma} = \{\gamma(t) \mid t \in [a, b]\}$. It is possible that different functions produce the same image, so it is often more useful to speak of $\mathit{\Gamma}$ than γ. Note the following terminology.

- $\gamma(a)$ is the initial point of γ.
- $\gamma(b)$ is the terminal point of γ.
- $\mathit{\Gamma}$ is simple if it "does not cross itself" i.e. $\nexists t_1 \in [a, b] \quad \nexists t_2 \in [a, b]$ such that $\gamma(t_1) = \gamma(t_2)$.
- γ is closed if $\gamma(a) = \gamma(b)$.
- γ is simple closed if it is both simple and closed.
- γ is smooth provided that γ' exists and is continuous.
- γ is a contour if it is piecewise smooth†.

If one path ends where another begins, it seems obvious that they can be added together to produce one long path. However, care must be taken in finding the domain of the new path.
Suppose that $\gamma_1\colon [a, b] \to \mathbb{C}$ and $\gamma_2\colon [c, d] \to \mathbb{C}$ are paths with $\gamma_1(b) = \gamma_2(c)$. In defining $\gamma_1 + \gamma_2$ it is required that the domain of γ_2 is translated so that it begins at b. Therefore

$$\begin{aligned}\gamma_1 + \gamma_2 \colon [a, b + |c - d|] &\to \mathbb{C} \\ t &\mapsto \begin{cases} \gamma_1(t) & \text{if } t \in [a, b] \\ \gamma_2(t + |c - d|) & \text{if } t \in [b, b + |c - d|] \end{cases}\end{aligned}$$

The orientation of a path can be reversed by defining $\gamma^*\colon [b, a] \to \mathbb{C}$.

Typically, a path may be expressed in the form $\gamma(t) = \gamma_u(t) + i\gamma_v(t)$ in which case the derivative is the obvious $\gamma'(t) = \gamma_u'(t) + i\gamma_v'(t)$.

(17.3.2) Path Integrals

A path integral is simply an integral taken along a path. By composing a path with a complex function a mapping of the form $\mathbb{R} \xrightarrow{\gamma} \mathbb{C} \xrightarrow{f} \mathbb{C}$ is produced which is a function from $\mathbb{R}$ to $\mathbb{C}$. Say $F(t)\colon \mathbb{R} \to \mathbb{C}$ with $F(t) = F_u(t) + iF_v(t)$, then

$$\int_a^b F \,\mathrm{d}t = \int_a^b F_u(t) \,\mathrm{d}t + i \int_a^b F_v(t) \,\mathrm{d}t$$

Now, where $z = \gamma(t)$, $\dfrac{\mathrm{d}z}{\mathrm{d}t} = \gamma'(t)$. Bearing this in mind,

$$\int_\gamma f(z) \,\mathrm{d}z \stackrel{\text{def}}{=} \int_a^b f(\gamma(t))\, \gamma'(t) \,\mathrm{d}t$$

Assertion 33 *Where f is a continuous complex function and γ_1 and γ_2 are contours with the terminal point of γ_1 coincidental with the initial point of γ_2,*

- $\int_{\gamma_1} f(z) \,\mathrm{d}z = -\int_{\gamma_1^*} f(z) \,\mathrm{d}z$.
- $\int_{\gamma_1+\gamma_2} f(z) \,\mathrm{d}z = \int_{\gamma_1} f(z) \,\mathrm{d}z + \int_{\gamma_2} f(z) \,\mathrm{d}z$

†Piecewise smooth means that the path is a sum of smooth paths

- *Where* $c \in \mathbb{C}$, $\int_{\gamma_1} cf(z)\ \mathrm{d}(z) = c \int_{\gamma_1} f(z)\ \mathrm{d}z$.

- *If* g *is another continuous function,* $\int_{\gamma_1} f(z) + g(z)\ \mathrm{d}z = \int_{\gamma_1} f(z)\ \mathrm{d}z + \int_{\gamma_1} g(z)\ \mathrm{d}z$.

Theorem 34 (Jordan Curve Theorem) *Suppose that* Γ *is a simple closed contour in* $\mathbb{C}$*. Then* Γ *divides* $\mathbb{C}$ *into two disjoint domains, one of which is bounded and the other of which is unbounded. They are denoted by* $\operatorname{Int}\Gamma$ *and* $\operatorname{Ext}\Gamma$ *respectively.*

Definition 35 *Let* f *be a complex function that is continuous on a domain* D*. Suppose that* F *is an analytic function with derivative* f *on* D*, then* F *is called an anti-derivative or primitive of* f*.*

An anti-derivative behaves as it would be expected to, since

$$\begin{aligned}\int_\gamma f(z)\ \mathrm{d}z &= \int_a^b f(\gamma(t))\gamma'(t)\ \mathrm{d}t \\ &= \int_a^b \frac{\mathrm{d}}{\mathrm{d}t} F(\gamma(t))\ \mathrm{d}t \\ &= [F(\gamma(t))]_a^b \\ &= F(\gamma(b)) - F(\gamma(a))\end{aligned}$$

So given an anti-derivative to a function, its contour integral can be readily evaluated by putting the end-points of the contour into the anti-derivative in this way. Furthermore, if the contour is closed then the integral is zero.

The ML Result

The ML result is concerned with finding bounds for integrals.

Definition 36 *Let* $\gamma\colon [a,b] \to \mathbb{C}$ *be a contour. The length of* γ *is defined as*

$$L(\gamma) = \int_a^b \left|\gamma'(t)\right|\ \mathrm{d}t$$

Lemma 37 *Suppose that* $\phi\colon [a,b] \to \mathbb{C}$ *is continuous, then*

$$\left|\int_a^b \phi(t)\ \mathrm{d}t\right| \leqslant \int_a^b |\phi(t)|\ \mathrm{d}t$$

Proof. Let $\int_a^b \phi(t)\ \mathrm{d}t = re^{i\theta}$,

$$\begin{aligned}\left|\int_a^b \phi(t)\ \mathrm{d}t\right| &= \left|re^{i\theta}\right| = |r| = r \\ r &= e^{-i\theta}\int_a^b \phi(t)\ \mathrm{d}t = \int_a^b e^{-i\theta}\phi(t)\ \mathrm{d}t \\ r &= \int_a^b \operatorname{Re}\left(e^{-i\theta}\phi(t)\right)\ \mathrm{d}t + i\int_a^b \operatorname{Im}\left(e^{-i\theta}\phi(t)\right)\ \mathrm{d}t\end{aligned}$$

Equating real and imaginary parts, it is evident that $\int_a^b \operatorname{Im}\left(e^{-i\theta}\phi(t)\right) \, \mathrm{d}t = 0$. Also, since $\operatorname{Re}(z) = x \leqslant \sqrt{x^2+y^2} = |z|$ this gives

$$\begin{aligned} r &\leqslant \int_a^b \left|e^{-i\theta}\phi(t)\right| \, \mathrm{d}t \\ &= \int_a^b \left|e^{-i\theta}\right| |\phi(t)| \, \mathrm{d}t \\ &= \int_a^b |\phi(t)| \, \mathrm{d}t \end{aligned}$$

Hence the result. □

Theorem 38 (The ***ML*** **Result)** *Suppose that f is continuous on the domain D and γ is a contour of length L. If $\exists M$ such that $|f(z)| \leqslant M$ for all z on Γ then $\left|\int_\gamma f(z) \, \mathrm{d}z\right| \leqslant ML$.*

Proof. Most of the work has already been done in Lemma 37,

$$\begin{aligned} \left|\int_\gamma f(z) \, \mathrm{d}z\right| &= \left|\int_a^b f(\gamma(t))\gamma'(t) \, \mathrm{d}t\right| \\ &\leqslant \int_a^b \left|f(\gamma(t))\gamma'(t)\right| \, \mathrm{d}t \quad \text{by Lemma 37} \\ &= \int_a^b |f(\gamma(t))| \left|\gamma'(t)\right| \, \mathrm{d}t \\ &\leqslant M \int_a^b \left|\gamma'(t)\right| \, \mathrm{d}t \\ &= ML \end{aligned}$$

Hence the result. □

Cauchy-Goursat Theorem

It was noted that the integral round a contour γ^* has the opposite sign to that round γ. In order to make the value of an integral well-defined it is conventional to traverse contours anticlockwise: The interior of a contour is on the left if the contour is simple closed.

Theorem 39 (Cauchy-Goursat) *Let f be a function that is analytic on a simply connected domain D. Then for all simple closed contours in D, $\int_\gamma f(z) \, \mathrm{d}z = 0$.*

This theorem is distinct from the result following Definition 35 in that it does not require the existence of an anti-derivative. A subtle point is that the domain must be simply connected: The result does not hold for $\frac{1}{z}$ on $\mathbb{C} \setminus \{0\}$ since this domain has a 'hole' in it at the origin. However, $\mathbb{C} \setminus \mathbb{R}_0^+$ would be a perfectly acceptable domain.

Since any non-simple contour can be thought of as a number of different 'loops' i.e. simple closed contours, it follows that if D is simply connected, f is analytic, and γ is any closed contour then $\int_\gamma f(z) \, \mathrm{d}z = 0$.

Suppose that γ_1 and γ_2 are contours in a simply connected domain D which share the initial point z_1 and share the terminal point z_2. The contour $\gamma = \gamma_1 + \gamma_2^*$ is then simple and closed and so $\int_{\gamma_1+\gamma_2^*} f(z) \, \mathrm{d}z = 0$. From this it is readily deduced that $\int_{\gamma_1} f(z) \, \mathrm{d}z = \int_{\gamma_2} f(z) \, \mathrm{d}z$.

When drawing diagrams it is common to see very oddly shaped contours‡. It would be incredibly difficult to describe these mathematically, so instead methods are sought to evaluate integrals round simpler

‡none here because they're difficult to create

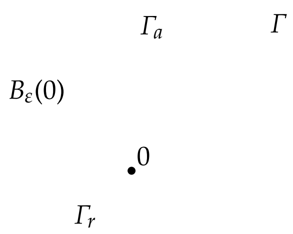

Figure 2: Contours around a 'bad' point

(preferable circular) contours.

Lemma 40 *If γ is any simple closed contour which has z_0 in its interior then $\int_\gamma \frac{1}{z-z_0}\,\mathrm{d}z = 2\pi i$.*

Proof. By Theorem 34 the interior of a contour is open, and so there exists $\varepsilon > 0$ such that $B_\varepsilon(z_0) \subset \operatorname{Int}\gamma$. Now consider $0 < r < \varepsilon$ and define $\gamma_r(t) = z_0 + re^{it}$ for $0 \leqslant t < 2\pi$. Let Γ_a join Γ to Γ_r. The situation is illustrated in Figure 2
The function $f(z) = \frac{1}{z-z_0}$ is analytic on the domain which is the interior of the curve

$$\Gamma_0 = \Gamma + \Gamma_a - \Gamma_r - \Gamma_a$$

Hence

$$\begin{aligned} \int_{\Gamma_0} \frac{1}{z-z_0}\,\mathrm{d}z &= 0 \\ 0 &= \int_\Gamma \frac{1}{z}\,\mathrm{d}z + \int_{\Gamma_a} \frac{1}{z-z_0}\,\mathrm{d}z - \int_{\Gamma_r} \frac{1}{z-z_0}\,\mathrm{d}z - \int_{\Gamma_a} \frac{1}{z-z_0}\,\mathrm{d}z \\ \int_\Gamma \frac{1}{z-z_0}\,\mathrm{d}z &= \int_{\Gamma_r} \frac{1}{z-z_0}\,\mathrm{d}z = \int_0^{2\pi} \frac{1}{re^{it}} ire^{it}\,\mathrm{d}t = 2\pi i \end{aligned} \tag{41}$$

Hence the integral round the complicated curve has the same value as the integral round the circle, which is readily seen to be $2\pi i$. □

The above result can be generalised to the function $\frac{1}{z-z_0}$ where $z_0 \in \operatorname{Int}\Gamma$.

Working along the same principle of summing round different contours, it is possible to extend Cauchy-Goursat.

Theorem 42 (Cauchy-Goursat For Multiply Connected Domains) *Suppose that Γ is simply closed contour in a simply connected domain D and that $\Gamma_1, \Gamma_2, \ldots, \Gamma_n$ are disjoint simple closed contours in $\operatorname{Int}\Gamma$ that aren't nested. If f is analytic on $D \setminus \bigcup_j \operatorname{Int}\Gamma_j$ then*

$$\int_\Gamma f(z)\,\mathrm{d}z = \sum_{j=1}^{n} \int_{\Gamma_j} f(z)\,\mathrm{d}z$$

Cauchy's Integral Theorems

Clearly complex integrals have some very desirable properties—they disappear a lot. Essentially what Theorem 39 says is that integrals round closed curves need only be evaluated round discontinuities in the domain of the integrand: Anywhere else the integral vanishes. Cauchy's integral formulae provide neat ways to evaluate these 'bad points' very easily.

Theorem 43 (Cauchy's Integral Formula) *Suppose that D is a simply connected domain upon which a complex function f is analytic. Let γ be a closed contour in D and $z_0 \in \operatorname{Int}\gamma$. Then*

$$f(z_0) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-z_0}\,\mathrm{d}z$$

Proof. First of all,

$$\begin{aligned}
\frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-z_0}\,\mathrm{d}z &= \frac{1}{2\pi i}\int_\gamma \frac{f(z)+f(z_0)-f(z_0)}{z-z_0}\,\mathrm{d}z \\
&= \frac{1}{2\pi i}\int_\gamma \frac{f(z)-f(z_0)}{z-z_0}\,\mathrm{d}z + \frac{1}{2\pi i}\int_\gamma \frac{f(z_0)}{z-z_0}\,\mathrm{d}z \\
&= \frac{1}{2\pi i}\int_\gamma \frac{f(z)-f(z_0)}{z-z_0}\,\mathrm{d}z + \frac{f(z_0)}{2\pi i}\int_\gamma \frac{1}{z-z_0}\,\mathrm{d}z \\
&= \frac{1}{2\pi i}\int_\gamma \frac{f(z)-f(z_0)}{z-z_0}\,\mathrm{d}z + f(z_0) \\
f(z_0) &= \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-z_0}\,\mathrm{d}z - \frac{1}{2\pi i}\int_\gamma \frac{f(z)-f(z_0)}{z-z_0}\,\mathrm{d}z
\end{aligned}$$

So it remains to show that $\frac{1}{2\pi i}\int_\gamma \frac{f(z)-f(z_0)}{z-z_0}\,\mathrm{d}z = 0$. By the Jordan Curve Theorem, Theorem 34, $B(z_0,\delta_1) \subset \operatorname{Int}\gamma$ and so for $0 < \alpha < \delta_1$ define the contour

$$\gamma_\alpha\colon [0,2\pi] \to B(z_0,\delta_1) \qquad \text{by} \quad t \mapsto z_0 + \alpha e^{it}$$

so clearly $\Gamma_\alpha \subset \operatorname{Int}\gamma$. Since the integrand $\frac{f(z)-f(z_0)}{z-z_0}$ is analytic on $D \setminus \operatorname{Int}\Gamma_\alpha$, by Cauchy Goursat for multiply connected domains,

$$\int_\gamma \frac{f(z)-f(z_0)}{z-z_0}\,\mathrm{d}z = \int_{\gamma_\alpha} \frac{f(z)-f(z_0)}{z-z_0}\,\mathrm{d}z$$

Hence it is required to show that

$$\lim_{\alpha\to 0}\left|\int_{\gamma_\alpha} \frac{f(z)-f(z_0)}{z-z_0}\,\mathrm{d}z\right| = 0$$

For this the ML result is used, and clearly $L = 2\pi\alpha$. To find M first of all note that

$$\lim_{z\to z_0} \frac{f(z)-f(z_0)}{z-z_0} = f'(z_0)$$

which can be written since f is analytic. Hence by considering $\varepsilon = 1$ in the definition of a limit,

$$\exists\delta_2 > 0 \text{ such that } 0 < |z-z_0| < \delta_2 \ \Rightarrow\ \left|\frac{f(z)-f(z_0)}{z-z_0} - f'(z_0)\right| < 1$$

However,

$$\begin{aligned}
\left|\frac{f(z)-f(z_0)}{z-z_0} - f'(z_0) + f'(z_0)\right| &\leqslant \left|\frac{f(z)-f(z_0)}{z-z_0} - f'(z_0)\right| + |f'(z_0)| \quad \text{by the triangle inequality} \\
&\leqslant 1 + |f'(z_0)|
\end{aligned}$$

Hence put $M = 1 + |f'(z_0)|$ so that for $\delta < \min(\delta_1,\delta_2)$

$$\left|\int_{\gamma_\alpha} \frac{f(z)-f(z_0)}{z-z_0}\,\mathrm{d}z\right| \leqslant \left(1+|f'(z_0)|\right)2\pi\alpha$$

Hence by the squeeze rule as $\alpha \to 0$ the required result is obtained. □

If a function has a number of bad points, then Cauchy-Goursat for multiply connected domains means that it is only necessary to evaluate an integral round each bad point and then sum. Cauchy's integral formula can be extended, since differentiating with respect to z_0 n times it is clear that

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_\gamma \frac{f(z)}{(z - z_0)^{n+1}} \, \mathrm{d}z$$

Note that if a function is analytic on a simply connected domain, then derivatives of every order exist and are analytic[§]. It has been seen that if a function is analytic then it integrates to zero round closed contours. However, the converse is also true.

Theorem 44 (Morera's Theorem) *Suppose a function f is continuous on a simply connected domain D, and that $\int_\gamma f(z) \, \mathrm{d}z = 0$ for all closed contours γ in D. Then f is analytic on D.*

Proof. Proof of Morera's Theorem is made by finding the anti-derivative of f, F. By showing that F is analytic, it is then concluded that its derivative f is analytic.

Suppose that z_1 and z_2 are distinct points in D. Since $\int_\gamma f(\zeta) \, \mathrm{d}\zeta = 0$ for all closed contours, it follows that the integrals along any contour joining z_1 and z_2 always have the same value.

Let a be some fixed point in D, then for some point z define the integral along any contour from a to z to be

$$F(z) = \int_a^z f(\zeta) \, \mathrm{d}\zeta$$

Now, $F(z) = \int_a^{z_0} f(\zeta) \, \mathrm{d}\zeta + \int_{z_0}^z f(\zeta) \, \mathrm{d}\zeta$. Hence

$$\frac{F(z) - F(z_0)}{z - z_0} = \frac{1}{z - z_0} \int_{z_0}^z f(\zeta) \, \mathrm{d}\zeta$$

So

$$\begin{aligned} \frac{F(z) - F(z_0)}{z - z_0} - f(z_0) &= \frac{1}{z - z_0} \int_{z_0}^z f(\zeta) \, \mathrm{d}\zeta - f(z_0) \, \mathrm{d}\zeta \\ &= \frac{1}{z - z_0} \int_{z_0}^z f(\zeta) - f(z_0) \, \mathrm{d}\zeta \end{aligned}$$

Now, since D is a domain, $\exists \delta_1 > 0$ such that $B(z_0, \delta_1) \subset D$, and the line connecting z_0 to z is in the ball if $0 < |z - z_0| < \delta_1$. This provides a bounding length for use in the ML result. Since any contour can be chosen the straight line is used.

$$\begin{aligned} \left| \frac{F(z) - F(z_0)}{z - z_0} - f(z_0) \right| &= \left| \frac{1}{z - z_0} \int_{z_0}^z f(\zeta) - f(z_0) \, \mathrm{d}\zeta \right| \\ &\leqslant \frac{|z - z_0|}{|z - z_0|} M = M \end{aligned}$$

What remains is to find M, the maximum value attained by $|f(z) - f(z_0)|$ on on the contour connecting z to z_0. However, by hypothesis f is continuous, and so from the definition of a limit,

$$\forall \varepsilon > 0 \; \exists \delta_2 > 0 \text{ such that } 0 < |\zeta - z_0| < \delta_2 \;\Rightarrow\; |f(\zeta) - f(z_0)| < \varepsilon$$

[§]This really should be presented ad proved as a proper theorem.

Hence for $\delta < \min(\delta_1, \delta_2)$

$$\left| \frac{F(z) - F(z_0)}{z - z_0} - f(z_0) \right| < \varepsilon$$

Hence

$$\lim_{z \to z_0} \left| \frac{F(z) - F(z_0)}{z - z_0} - f(z_0) \right| = 0$$

Since z_0 was chosen arbitrarily it follows that F is differentiable everywhere on $\mathbb{C}$ i.e. is analytic. Furthermore, the derivative of F is f, which, since it is the derivative of an analytic function must in turn be analytic. Hence the result. □

(17.4) Properties & Uses Of Complex Functions

(17.4.1) Series Expansion Of Complex Functions

Sequences Of Functions

In a similar way to real analysis, sequences can be defined such as

$$\{z_n\} = \frac{2 + \sin n}{n}$$

which is clearly convergent to zero. However, another parameter, $z \in \mathbb{C}$, can be introduced. The convergence of the sequence is then dependent upon what values the parameter takes. A new function f can then be defined as $f(z) = \lim_{n \to \infty} f_n(z)$.

Definition 45 *1. Let $\{z\}$ be a sequence.*

$$\lim_{n \to \infty} z_n = w \iff \forall \varepsilon > 0 \; \exists N \in \mathbb{N} \text{ such that } |z_n - w| < \varepsilon \text{ for } n > N$$

2. *For a sequence $\{f_n(z)\}$ which is convergent to $f(z)$,*
   (a) *If $\lim_{n \to \infty} f_n(z) = f(z)$ only for some $z \in U \subset \mathbb{C}$, then $\{f_n(z)\}$ is said to be piecewise convergent. Note that the convergence depends on z.*
   (b) *If $\lim_{n \to \infty} f_n(z) = f(z)$ for all $z \in \mathbb{C}$ then $\{f_n(z)\}$ is uniformly convergent. Note that the convergence does not depend on z.*

Assertion 46 *The following theorems are asserted without proof.*

i. *If a sequence $\{f_n(z)\}$ is uniformly convergent to f and each $f_n(z)$ is continuous, then f is continuous.*
ii. *Let $f_n \colon [a, b] \to \mathbb{C}$ for all $n \in \mathbb{N}$. If the sequence of these functions is uniformly convergent, then*

$$\lim_{n \to \infty} \int_a^b f_n(t) \, \mathrm{d}t = \int_a^b f(t) \, \mathrm{d}t$$

It follows that if Γ is a contour in some subset of $\mathbb{C}$ and $\{f_n(z)\}$ is uniformly convergent to f then

$$\lim_{n \to \infty} \int_\Gamma f_n(t) \, \mathrm{d}t = \int_\Gamma f(t) \, \mathrm{d}t$$

Furthermore, if each f_n is analytic on the same domain, then f is analytic on that domain. (This is shown from Cauchy-Goursat and Morera's Theorem).

Consider $\sum_{k=0}^{\infty} f_k(z)$, then its convergence can be determined by considering the sequence of nth partial sums, $a_n(z) = \sum_{k=0}^{n} f_k(z)$. Note that $\sum_{k=0}^{\infty} f_k(z)$ is convergent whenever $\{a_n\}$ is convergent, and they are convergent in

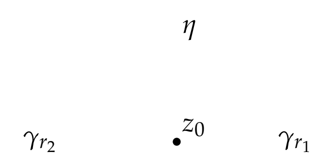

Figure 3: Contours in an annulus, used for proving Laurent's Theorem.

the same way i.e. uniform or piecewise.

Theorem 47 (Weierstrass M Test) *Let $\sum_{k=0}^{\infty} M_k$ be a convergent series of positive terms then $\sum_{k=0}^{\infty} f_k(z)$ is uniformly convergent on a region R provided that $|f_k(z)| < M_k$ for all $z \in R$.*

Theorem 48 *If $\sum_{k=0}^{\infty} f_k(z)$ is uniformly convergent to $f(z)$ on a region R then*

(i) If each f_k is continuous, then so is f.

(ii) If each f_k is analytic, then so is f and $\dfrac{\mathrm{d}f}{\mathrm{d}z} = \sum_{k=0}^{\infty} \dfrac{\mathrm{d}}{\mathrm{d}z} f_k(z)$.

(iii) For any contour Γ in R, $\int_{\Gamma} f(z)\, \mathrm{d}z = \sum_{k=0}^{\infty} \int_{\Gamma} f_k(z)\, \mathrm{d}z$.

Recall that $\sum_{n=-\infty}^{\infty} a_n(z-z_0)^n$ is a power series about the point z_0. If $R = \lim_{n\to\infty} \dfrac{|a_n|}{|a_{n+1}|}$ then R is the radius of convergence i.e. the sum is convergent for $|z - z_0| < R$.

Expansions Of A Complex Function

Theorem 49 (Laurent's Theorem) *Suppose that f is analytic on an annulus D centred at z_0, $R_1 < |z - z_0| < R_2$, and let γ_r be a contour of radius r centered at z_0 such that $R_1 < r < R_2$. Then for $z \in \operatorname{Int} D$,*

$$f(z) = \sum_{n=-\infty}^{\infty} c_n(z-z_0)^n \quad \textit{where} \quad c_n = \frac{1}{2\pi i} \int_{\gamma_r} \frac{f(s)}{(s-z_0)^{n+1}}\, \mathrm{d}s$$

Proof. Suppose $z \in \operatorname{Int} D$ and choose r_1, r_2 such that $R_1 < r_1 < |z - z_0| < r_2 < R_2$ and let γ_{r_1} and γ_{r_2} be the corresponding contours centred at z_0, as shown in Figure 13.

Now define the contour

$$\Gamma = \gamma_{r_2} + \eta - \gamma_{r_1} - \eta$$

By hypothesis f is analytic on $\Gamma \cup \operatorname{Int} \Gamma$, which is the region shown shaded in Figure 13. Hence Cauchy's integral formula can be used to give

$$
\begin{aligned}
f(z) &= \frac{1}{2\pi i}\int_{\Gamma}\frac{f(s)}{s-z}\,\mathrm{d}s \\
&= \frac{1}{2\pi i}\left(\int_{\gamma_{r_1}}\frac{f(s)}{s-z}\,\mathrm{d}s + \int_{\eta}\frac{f(s)}{s-z}\,\mathrm{d}s - \int_{\gamma_{r_2}}\frac{f(s)}{s-z}\,\mathrm{d}s - \int_{\eta}\frac{f(s)}{s-z}\,\mathrm{d}s\right) \\
&= \frac{1}{2\pi i}\left(\int_{\gamma_{r_1}}\frac{f(s)}{s-z}\,\mathrm{d}s + \int_{\gamma_{r_2}}\frac{f(s)}{z-s}\,\mathrm{d}s\right) \qquad (50)
\end{aligned}
$$

Now,

$$
\begin{aligned}
\frac{1}{s-z} &= \frac{1}{(s-z_0)-(z-z_0)} \\
&= \frac{1}{s-z_0}\frac{1}{1-\frac{z-z_0}{s-z_0}} \\
&= \frac{1}{s-z_0}\sum_{n=0}^{\infty}\left(\frac{z-z_0}{s-z_0}\right)^n \quad \text{for} \quad \left|\frac{z-z_0}{s-z_0}\right|<1 \\
\text{and similarly} \quad \frac{1}{z-s} &= \frac{1}{z-z_0}\sum_{n=0}^{\infty}\left(\frac{s-z_0}{z-z_0}\right)^n \quad \text{for} \quad \left|\frac{s-z_0}{z-z_0}\right|<1
\end{aligned}
$$

Using these expansions to substitute into equation (50) gives

$$
f(z) = \frac{1}{2\pi i}\left(\int_{\gamma_{r_2}}\sum_{n=0}^{\infty}\frac{f(s)}{s-z_0}\left(\frac{z-z_0}{s-z_0}\right)^n\,\mathrm{d}s + \int_{\gamma_{r_1}}\sum_{n=0}^{\infty}\frac{f(s)}{z-z_0}\left(\frac{s-z_0}{z-z_0}\right)^n\,\mathrm{d}s\right)
$$

Now, the integral and the sum can only be 'swapped round' if the sum is uniformly convergent on the region under consideration. Now, for the first summation, the integration is over γ_{r_2} so $|s-z_0| = r_2$.

$$
\begin{aligned}
\left|\frac{f(s)}{z-z_0}\left(\frac{s-z_0}{z-z_0}\right)^n\right| &= |f(s)|\frac{|(z-z_0)^n|}{|(s-z_0)^{n+1}|} \\
&\leqslant M_1\frac{|(z-z_0)^n|}{r_2^{n+1}}
\end{aligned}
$$

Where M_1 is the maximum value of $f(s)$ along γ_{r_2}. Also, since $z \in \operatorname{Int}\Gamma$ it must be the case that $|z-z_0| < r_2$. This produces a series $\sum_{n=0}^{\infty} M_1\frac{|z-z_0|^n}{r_2^{n+1}}$ which consists of only positive terms and is convergent. Hence by the Weierstrass M test the first summation is uniformly convergent on $\Gamma \cup \operatorname{Int}\Gamma$.

For the second summation, consider the mapping $n \mapsto -n$. An application of precisely the same argument shows that this is also uniformly convergent on $\Gamma \cup \operatorname{Int}\Gamma$ and hence

$$
f(z) = \frac{1}{2\pi i}\left(\sum_{n=0}^{\infty}\int_{\gamma_{r_2}}\frac{f(s)}{(s-z_0)^{n+1}}(z-z_0)^n\,\mathrm{d}s + \sum_{n=-\infty}^{-1}\int_{\gamma_{r_1}}\frac{f(s)}{(s-z_0)^{n+1}}(z-z_0)^n\,\mathrm{d}s\right)
$$

Now, consider $r_1 < r < r_2$, then by Cauchy-Goursat for multipally connected domains (wich is used 'backwards'),

$$
\begin{aligned}
f(z) &= \frac{1}{2\pi i}\sum_{n=-\infty}^{\infty}\int_{\gamma_r}\frac{f(s)}{(s-z_0)^{n+1}}(z-z_0)^n\,\mathrm{d}s \\
&= \sum_{-\infty}^{\infty} c_n(z-z_0)^n \quad \text{where } c_n = \frac{1}{2\pi i}\int_{\gamma_r}\frac{f(s)}{(s-z_0)^{n+1}}\,\mathrm{d}s
\end{aligned}
$$

Hence the result. $\square$

In practice not all of the terms need be present. Indeed, a Laurent series is rather like a Taylor series but starting 'further down'. This different start position is caused by having to take into account a bad point in the domain and so creating an annulus. If there is no such bad point to cope with, a normal Taylor series suffices as the power series expansion.

Theorem 51 (Taylor's Theorem) *Suppose that f is an analytic function on the domain $B = B(z_0, R)$, then for $z \in B$,*

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n$$

Proof. Evaluating at z_0, the theorem clearly holds. Hence consider the annulus $B_0 = B \setminus \{z_0\}$, then applying Laurent's Theorem,

$$f(z) = \sum_{n=-\infty}^{\infty} c_n(z - z_0)^n \quad \text{where } c_n = \frac{1}{2\pi i}\int_{\gamma_r} \frac{f(s)}{(s - z_0)^{n+1}}\,\mathrm{d}s$$

Now, considering the different possibilities for n,

- if $n > 0$ then $c_n = \dfrac{f^{(n)}(z_0)}{n!}$ by Cauchy's integral formula for derivatives.
- if $n = 0$ then $c_0 = f(z_0)$ by Cauchy's integral formula.
- if $n < 0$ then $\dfrac{f(s)}{(s - z_0)^{n+1}} = f(s)(s - z_0)^p$ where $p > 0$. On B, c_n is now the integral of an analytic function round a closed contour in a simply connected domain, and hence is zero.

Hence the result. $\square$

It can be shown that the Laurent or Taylor expansion of a function is unique on any given annulus or domain. This means that changing the subset of $\mathbb{C}$ upon which the expansion is considered may well change the expansion itself. This is best illustrated by means of an example.

Example 52 *Find all the Laurent expansions of the function $f(z) = \dfrac{1}{z(z-2)}$ centred at 0.*

Proof. Solution There are two Laurent expansions of this function, on the annuli

$$D_1\colon 0 < |z| < 2 \quad \text{and} \quad D_2\colon |z| > 2$$

Taking each case in turn,

on D_1: Using partial fractions,

$$f(z) = \frac{\frac{-1}{2}}{z} - \frac{1}{4}\frac{1}{1 - \frac{1}{2}z}$$

Now, since $|z| < 2$, $\left|\frac{1}{2}z\right| < 1$ and hence

$$\begin{aligned} f(z) &= \frac{\frac{-1}{2}}{z} - \frac{1}{4}\sum_{n=0}^{\infty}\left(\frac{1}{2}z\right)^n \\ &= \frac{\frac{-1}{2}}{z} - \sum_{n=0}^{\infty}\frac{z^n}{2^{n+2}} \\ &= -\sum_{n=-1}^{\infty}\frac{z^n}{2^{n+2}} \end{aligned}$$

This is therefore the unique Laurent series expansion on D_1.

on D_2: Again using partial fractions,

$$
\begin{aligned}
f(z) &= \frac{\frac{-1}{2}}{z} + \frac{\frac{1}{2}}{z-2} \\
&= \frac{\frac{-1}{2}}{z} + \frac{\frac{1}{2}}{z\left(1-\frac{2}{z}\right)} \\
&= \frac{\frac{-1}{2}}{z} + \frac{\frac{-1}{2}}{z}\sum_{n=0}^{\infty}\left(\frac{2}{z}\right)^n \\
&= \frac{\frac{-1}{2}}{z}\sum_{n=1}^{\infty}\left(\frac{2}{z}\right)^n \\
&= \sum_{n=1}^{\infty}\frac{2^{n-1}}{z^{n+1}} \\
&= \sum_{n=-\infty}^{-2}\frac{1}{2^{n-2}}z^n
\end{aligned}
$$

Both cases are now covered, each of them has its own unique Laurent series. □

(17.4.2) Singularities & Residues

Singularities

It has been seen that functions may have 'bad points' in their domain which prevent—or in some cases cause—certain results to be used. This idea is now formalised.

For an analytic function f, p is a zero of f if $f(p) = 0$. Consider the Taylor series expansion of such a function about a zero p,

$$f(z) = \sum_{n=0}^{\infty} a_n (z-p)^n$$

If $f(z) \neq 0$ then the sum must have a first non-zero term, say when $n = N$. This gives $f^{(N)}(p) \neq 0$ and $f^{(m)}(p) = 0$ for all $m < N$. Here N is called the order of the zero—the number of times the function can be differentiated without it disappearing. Observe that the Taylor series can be expressed as

$$f(z) = (z-p)^N \left(\frac{f^{(N)}(p)}{N!} + \frac{f^{(N+1)}(p)}{(N+1)!}(z-p) + \dots\right)$$

Which is in the form $(z-p)^N g(z)$ where g is an analytic function.

Definition 53 *Suppose that $p \in \mathbb{C}$ and f is a function. p is an isolated singularity of f if*

- *f is not analytic at p.*
- *$\exists \varepsilon > 0$ such that f is analytic on $B(p, \varepsilon) \setminus \{p\}$.*

Notice that if there is an isolated singularity at p then f has a Laurent series expansion about p, although the radius of convergence may only be very small, say

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z-p)^n = \sum_{n=1}^{\infty} \frac{b_n}{(z-p)^n} + \sum_{n=0}^{\infty} a_n (z-p)^n$$

where $b_n = c_{-n}$.

The term $\displaystyle\sum_{n=1}^{\infty} \frac{b_n}{(z-p)^n}$ is called the principal part of f.

- If there are a finite number of terms in the principal part of f, i.e. $\exists N \in \mathbb{N}$ such that $b_n = 0\ \forall n > N$, then p is said to be a pole of order N. If $N = 1$ then p is a simple pole.
- If there does not exist such an N, so that $\forall n \in \mathbb{N}\ \exists k > n$ such that $b_k \neq 0$, then p is called an isolated essential singularity.
- If $b_n = 0\ \forall n \in \mathbb{N}$ then p is called a removable singularity.

These different situations are probably best illustrated by means of the following three cases.

- Let $f(z) = \dfrac{1}{z}$. This is already its Laurent expansion about zero, and clearly $c_n \neq 0 \Leftrightarrow n = -1$. Hence 0 is a simple pole of f.
- Let $f(z) = \dfrac{z+3}{(z-2)^5}$. Performing some algebra,

$$f(z) = \frac{z+3}{(z-2)^5} = \frac{z-2+5}{(z-2)^5} = \frac{1}{(z-2)^4} + \frac{5}{(z-2)^5}$$

  From this Laurent expansion it is evident that f has a pole of order 5 at $z = 2$.
- The function $f(z) = \dfrac{z^7}{z^2}$ is not defined at $z = 0$. However, clearly its Laurent expansion is z^5 and so it has a removable singularity at $z = 0$. A more usual example of this is the function $\sin \dfrac{1}{z}$.

Theorem 54 *The following are equivalent*

1. *$f(z)$ has a pole of order N.*
2. $\lim_{z\to p} (z-p)^{N+1} f(z) = 0$.
3. *$(z-p)^N f(z)$ has a removable singularity.*

Proof. To prove equivalence a circular relationship is established.

$1 \Rightarrow 2$ Consider the Laurent expansion of f,

$$f(z) = \frac{b_N}{(z-p)^N} + \frac{b_{N-1}}{(z-p)^{N-1}} + \cdots + a_0 + a_1(z-p) + \ldots$$
$$(z-p)^{N+1} f(z) = \frac{b_N(z-p)^{N+1}}{(z-p)^N} + \frac{b_{N-1}(z-p)^{N-1}}{(z-p)^{N-1}} + \cdots + a_0(z-p)^{N+1} + a_1(z-p)^{N+2} + \ldots$$

From this it is clear that $\lim_{z\to p} (z-p)^{N+1} f(z) = 0$, as required.

$2 \Rightarrow 3$

$$\lim_{z\to p} (z-p)^{N+1} f(z) = 0$$
$$\lim_{z\to p} (z-p)^{N+1} \sum_{n=-\infty}^{\infty} c_n (z-p)^n = 0$$
$$\lim_{z\to p} \sum_{n=-\infty}^{\infty} c_n (z-p)^{n+N+1} = 0$$
$$\lim_{z\to p} \sum_{(-(N+1)}^{\infty} c_n (z-p)^{n+N+1} = 0 \quad \text{by excluding bad terms}$$
$$\lim_{z\to p} \sum_{m=0}^{\infty} c_m (z-p)^{m} = 0 \quad \text{by putting } m = n - (N+1)$$

This series has no terms of negative power, so the singularity must be removable.

$3 \Rightarrow 1$ In a similar way to before, since $(z-p)^N f(z)$ has a removable singularity,

$$\begin{aligned}
(z-p)^N f(z) &= \sum_{n=0}^{\infty} a_n (z-p)^n \\
f(z) &= \sum_{n=0}^{\infty} a_n \frac{(z-p)^n}{(z-p)^N} \\
&= \sum_{m=1}^{N} \frac{b_m}{(z-p)^m} + \sum_{m=0}^{\infty} a_m (z-p)^m
\end{aligned}$$

With the last line following by shifting the sum. By definition it is evident that f has a pole of order N, hence this implication is proved.

It has now been established that $1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 1$ so the equivalence is proved. □

Theorem 55 (The Great Picard) *Suppose that f is a complex function with an isolated essential singularity at z_0. Then f takes all values with at most one exception, in any neighbourhood of z_0.*

This theorem indicates the very different nature of simple poles and isolated essential singularities. If z_0 is a simple pole of f then $|f(z)| \to \infty$ as $z \to z_0$ without exception. With the isolated essential singularities, $f(z)$ can approach any value, depending on how $z \to z_0$.

Residues

Definition 56 *Let f be a complex function defined on a domain D, and let p be an isolated singularity in D. If the Laurent expansion of f about p is*

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z-p)^n$$

then the residue of f at p is defined as

$$\operatorname{Res}(f,p) \stackrel{def}{=} c_{-1} = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{(z-p)^{-1+1}}\, \mathrm{d}z = \frac{1}{2\pi i} \int_\gamma f(z)\, \mathrm{d}z$$

The integral expression follows from the difinition of a Laurent series.

Theorem 57 (The Residue Theorem) *Suppose that γ is a simple closed contour in a domain D, and let f be a complex function which is analytic on D except at finitely many points, $p_1, p_2, \ldots, p_k$, all of which line in $\operatorname{Int}\gamma$. Then*

$$\int_\gamma f(z)\, \mathrm{d}z = 2\pi i \sum_{i=1}^{k} \operatorname{Res}(f, p_i)$$

Proof. Since the singularities are isolated they can each be contained within a contour, $\gamma_1, \gamma_2, \ldots, \gamma_k$, say. By Cauchy-Goursat for multiply connected domains (Theorem 42),

$$\begin{aligned}
\int_\gamma f(z)\, \mathrm{d}z &= \sum_{i=1}^{k} \int_{\gamma_i} f(z)\, \mathrm{d}z \\
&= 2\pi i \sum_{i=1}^{k} \operatorname{Res}(f, p_i)
\end{aligned}$$

which follows directly from the definition of a residue. □

The obvious way to calculate a residue is to find the appropriate Laurent expansion and pick out the -1th coefficient. However, there are more efficient ways to proceed.

Rule 1. If f has a simple pole at p, $\operatorname{Res}(f, p) = \lim_{z \to p} (z - p) f(z)$.

Rule 2. If $f(z) = \dfrac{g(z)}{h(z)}$ where g and h are analytic on a domain U, then if f has a simple pole at p and $g(p) = 0$ then $\operatorname{Res}(f, p) = \lim_{z \to p} \dfrac{h(z)}{g'(z)}$.

Rule 3. If f has a pole of order m at p, then where $g(z) = (z - p)^m f(z)$, $\operatorname{Res}(f, p) = \lim_{z \to p} \dfrac{g^{(m-1)}(z)}{(m-1)!}$.

Rule 4. If $f(z) = \dfrac{g(z)}{h(z)}$ where g and h are analytic on a domain U, then if f has an isolated essential singularity at p calculate the first few terms of the Taylor series of g and h, and then divide.

Rule 5. Calculate the Laurent series expansion of f directly.

(17.4.3) Evaluation Of Real Integrals

Integrals Of Trigonometric Formulae

An integral of the form $\int_0^{2\pi} f(\cos\theta,\ \sin\theta)\ \mathrm{d}\theta$ can be evaluated by making a substitution—or rather the opposite to a substitution. Consider $z \mapsto e^{i\theta}$ so

$$\cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right) \qquad \text{and} \qquad \sin\theta = \frac{1}{2i}\left(z - \frac{1}{z}\right)$$

This substitution also gives $\dfrac{\mathrm{d}z}{\mathrm{d}\theta} = iz$. Hence

$$\int_0^{2\pi} f(\cos\theta,\ \sin\theta)\ \mathrm{d}\theta = \int_\gamma f\left(\frac{1}{2}\left(z + \frac{1}{z}\right)\ ,\ \frac{1}{2i}\left(z - \frac{1}{z}\right)\right) \frac{1}{iz}\ \mathrm{d}z$$

where γ is the unit circle, and note that on the unit circle $z = \bar{z}$. Finding and evaluating residues at the singularities in $\operatorname{Int}\gamma$ allows the integral to be evaluated using Theorem 57.

It is necessary to integrate the complex integral round the unit circle so that it reduces to the original integral: introducing a factor to change the radius will change the value of the integral.

Improper Integrals

Improper integrals of the variety with infinite limits can be evaluated, provided the Cauchy principal value definition of the integral is taken, i.e.

$$\int_{-\infty}^{\infty} f(x)\ \mathrm{d}x \stackrel{\text{def}}{=} \lim_{R \to \infty} \int_{-R}^{R} f(x)\ \mathrm{d}x$$

This definition has the advantage that when f is an odd function the value of the integral is zero at all stages of the limiting progress. Indeed, other definitions produce different values for the integral.

Simply substituting z for x in f, consider the integral of $f(z)$ round the semi-circular contour

$$\gamma_R = \{z \mid |z| = r \text{ then } \operatorname{Im} z = 0\}$$

This is illustrated in Figure 17.4.3.

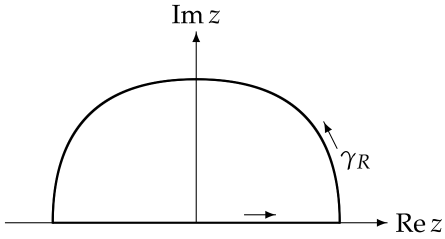

Figure 4: Contour of integration for use in evaluating improper real integrals.

It is required that $f(z)$ is analytic on the upper half plane, except at finitely many poles, none of which are on the real axis. R is then chosen so that all the singularities of $f(z)$ (in the upper half plane) are contained in Int γ_R. The problem is now reduced to finding the residues of $f(z)$ at the singularities in Int γ_R and using the residue theorem (Theorem 57). Where γ_{R_c} is the circular part of γ_R and γ_{R_s} is the straight part, this gives

$$\begin{aligned}
\lim_{R\to\infty}\left(\int_{\gamma_R} f(z)\,\mathrm{d}z\right) &= \lim_{R\to\infty}\left(\int_{\gamma_{R_c}} f(z)\,\mathrm{d}z + \int_{\gamma_{R_s}} f(z)\,\mathrm{d}z\right) \\
&= \int_{-\infty}^{\infty} f(x)\,\mathrm{d}x + \lim_{R\to\infty}\int_{\gamma_{R_c}} f(z)\,\mathrm{d}z \\
\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x &= \lim_{R\to\infty}\left(\int_{\gamma_R} f(z)\,\mathrm{d}z - \int_{\gamma_{R_c}} f(z)\,\mathrm{d}z\right) \\
&= 2\pi i \sum \operatorname{Res}(f,p) - \lim_{R\to\infty}\int_{\gamma_{R_c}} f(z)\,\mathrm{d}z
\end{aligned}$$

The remaining problem now is to show that the integral round the semi-circular part of γ_R is zero.

In some cases it is possible to use the ML result, observe that on γ_{R_c}, $|z| = R$ and that

$$|z| = |z - c + c| \leqslant |z + c| - |c| \quad \text{hence} \quad |z + c| \geqslant |z| + |c|$$

In other cases a more detailed analysis is needed.

Lemma 58 (Jordan's Lemma) *Let $\gamma_R = \left\{z \mid z = Re^{i\theta} \text{ for } 0 \leqslant \theta \leqslant \pi\right\}$ and suppose that $M(R) = \sup_{z\in\gamma_R} |f(z)|$. Then if $\lim_{R\to\infty} M(R) = 0$,*

$$\lim_{R\to\infty}\int_{\gamma_R} e^{i\alpha z} f(z)\,\mathrm{d}z = 0$$

for all $\alpha > 0$.

This allows integrals of the form

$$\int_{-\infty}^{\infty} \frac{\cos x}{p(x)}\,\mathrm{d}x \quad p(x) \text{ is a polynomial}$$

to be solved by considering $\dfrac{e^{iz}}{p(z)}$.

Indenting Integrals

If there are any singularities on the real axis, then the method described above does not work. In order to evaluate such integrals the singularities are 'jumped over'. The contour of integration is shown in Figure 17.4.3

It is assumed that each singularity is a simple pole. Round each pole p define the semi-circular contour of

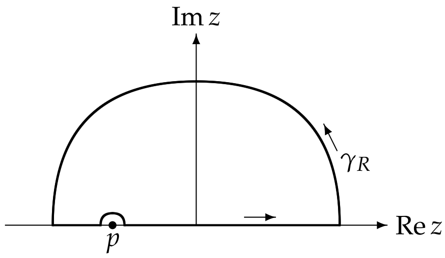

Figure 5: Contour of integration for use in evaluating improper real integrals with singularities.

radius ε, γ_ε, which is traversed clockwise—the wrong way. Hence

$$\int_\Gamma f(z)\,\mathrm{d}z = \int_{-R}^{p-\varepsilon} f(z)\,\mathrm{d}z + \int_{\gamma_\varepsilon} f(z)\,\mathrm{d}z + \int_{p+\varepsilon}^{R} f(z)\,\mathrm{d}z + \int_\Gamma f(z)\,\mathrm{d}z$$

Now, since each pole is simple, the laurent expansion of $f(z)$ about p is of the form

$$f(z) = \frac{\mathrm{Res}\,(f,p)}{z-p} + g(z)$$

where g is analytic and so is continuous, hence by the ML result

$$\lim_{\varepsilon\to 0}\int_{\gamma_\varepsilon} g(z)\,\mathrm{d}z = 0$$

and hence

$$\lim_{\varepsilon\to 0}\int_{\gamma_\varepsilon} f(z)\,\mathrm{d}z = \lim_{\varepsilon\to\infty}\int_{\gamma_\varepsilon} \frac{\mathrm{Res}\,(f,p)}{z-p}\,\mathrm{d}z = -\pi i\mathrm{Res}\,(f,p)$$

with the minus because γ_ε is being traversed clockwise, and πi instead of $2\pi i$ (from Cauchy's integral formula) because only half of γ_ε is being traversed.

Hence where Γ is the complete contour,

$$\begin{aligned}
\int_\Gamma f(z)\,\mathrm{d}z &= \lim_{\varepsilon\to 0}\left(\int_{-R}^{p-\varepsilon} f(z)\,\mathrm{d}z + \int_{\gamma_\varepsilon} f(z)\,\mathrm{d}z + \int_{p+\varepsilon}^{R} f(z)\,\mathrm{d}z + \int_{\Gamma_c} f(z)\,\mathrm{d}z\right)\\
2\pi i\sum_{p\in\mathrm{Int}\,\Gamma}\mathrm{Res}\,(f,p) &= \int_{-R}^{R} f(z)\,\mathrm{d}z + \int_{\Gamma_c} f(z)\,\mathrm{d}z - \pi i\sum_{p\in\mathbb{R}}\mathrm{Res}\,(f,p)\\
\int_{-R}^{R} f(z)\,\mathrm{d}z &= 2\pi i\sum_{p\in\mathrm{Int}\,\Gamma}\mathrm{Res}\,(f,p) + \pi i\sum_{p\in\mathbb{R}}\mathrm{Res}\,(f,p) - \int_{\Gamma_c} f(z)\,\mathrm{d}z
\end{aligned}$$

Using Jordan's lemma to show that the integral round the semi-circular contour γ_c is zero as $R\to\infty$ the solution to the original integral can be found.