Chapter 24

MSMYA1 Partial Differential Equations

(24.1) Analytical Techniques

(24.1.1) Common Equations & Equation Classification

Definition 1 *A partial differential equation is a relationship of the form*

$$f\left(\mathbf{x}, \mathbf{u}(\mathbf{x}), \frac{\partial \mathbf{u}}{\partial x_i}, \ldots, \frac{\partial^n \mathbf{u}}{\partial x_{i_1} \partial x_{i_2} \ldots \partial x_{i_n}}\right) = 0$$

where $\mathbf{u}$ *is a vector of dependent variables and* $\mathbf{x}$ *is a vector of the independent variables. The order of the equation is the order of the highest order derivative.*

Definition 2 *A partial differential equation is linear if the dependent variables and their derivatives appear only linearly.*

Definition 3 *A partial differential equation is quasi-linear if the derivatives of the dependent variables appear linearly.*

Definition 4 *A partial differential equation is homogeneous if* $\mathbf{u} = \mathbf{0}$ *is a solution.*

Some common differential equations and their classifications are listed in table 1.

Name	Equation	Classification
Laplace's Equation	$\nabla^2\phi = 0$	Linear, homogeneous
The Wave Equation	$\nabla^2\phi = \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2}$	Linear
The Diffusion Equation	$D\nabla^2\phi = \frac{\partial\phi}{\partial t}$	Linear
The KdV Equation	$\frac{\partial\phi}{\partial t} + \phi\frac{\partial\phi}{\partial x} + \frac{\partial^3\phi}{\partial x^3} = 0$	Quasi-linear, homogeneous

Table 1: Some common PDEs and their classifications

It is usual to solve PDEs over some kind of solution domain, which could be finite or infinite. If this domain is D then it is usual to specify conditions on its boundary, ∂D.

- Dirichlet boundary conditions specify $\phi(\mathbf{x})$ on ∂D.
- Neumann boundary conditions specify the gradient of the normal vector on ∂D i.e. give the value of $\frac{\partial\phi}{\partial\mathbf{n}} = \nabla\phi \cdot \mathbf{n}$ on ∂D.

As well as boundary conditions, initial conditions may be specified where ϕ depends on time t.

The method of separation of variables has already been covered in Chapter **??**. Other analytical methods for solving PDEs are now discussed.

(24.1.2) D'Allembert's Solution Of The 1 Dimensional Wave Equation

The one dimensional wave equation can be used to model vibrations in a string of finite or infinite length. The problem is to solve

$$\frac{\partial y}{\partial x} = \frac{1}{c^2}\frac{\partial y}{\partial t} \tag{5}$$

subject to the initial conditions $y = y_0(x)$ and $\frac{\partial y}{\partial t} = v_0(x)$ when $t = 0$.

Introduce the new variables $\xi = x - ct$ and $\eta = x + ct$ then by the Chain rule

$$\begin{aligned} \frac{\partial}{\partial x} &= \frac{\partial \xi}{\partial x}\frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial x}\frac{\partial}{\partial \eta} & \qquad \frac{\partial}{\partial t} &= \frac{\partial \xi}{\partial t}\frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial t}\frac{\partial}{\partial \eta} \\ &= \frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta} & &= -c\frac{\partial}{\partial \xi} + c\frac{\partial}{\partial \eta} \end{aligned}$$

This gives

$$\begin{aligned} \frac{\partial^2 y}{\partial x^2} &= \left(\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}\right)\left(\frac{\partial y}{\partial \xi} + \frac{\partial y}{\partial \eta}\right) \\ &= \frac{\partial^2 y}{\partial \xi^2} + 2\frac{\partial^2 y}{\partial \xi \partial \eta} + \frac{\partial^2 y}{\partial \eta^2} \\ \text{and } \frac{\partial^2 y}{\partial t^2} &= \left(-c\frac{\partial}{\partial \xi} + c\frac{\partial}{\partial \eta}\right)\left(-c\frac{\partial y}{\partial \xi} + c\frac{\partial y}{\partial \eta}\right) \\ &= c^2\frac{\partial^2 y}{\partial \xi^2} + 2c^2\frac{\partial^2 y}{\partial \xi \partial \eta} + c^2\frac{\partial^2 y}{\partial \eta^2} \end{aligned}$$

Now substituting into (5) gives

$$\begin{aligned} \frac{\partial^2 y}{\partial \xi^2} + 2\frac{\partial^2 y}{\partial \xi \partial \eta} + \frac{\partial^2 y}{\partial t^2} &= \frac{1}{c^2}\left(c^2\frac{\partial^2 y}{\partial \xi^2} + 2c^2\frac{\partial^2 y}{\partial \xi \partial \eta} + c^2\frac{\partial^2 y}{\partial \eta^2}\right) \\ \frac{\partial^2 y}{\partial \xi \partial \eta} &= 0 \\ \text{integrating wrt } \eta,\ \frac{\partial y}{\partial \xi} &= \overline{f}(\xi) \\ \text{integrating wrt } \eta,\ y &= \int_c^{\xi} \overline{f}(s)\,\mathrm{d}s + g(\eta) \\ &= f(\xi) + g(\eta) \\ &= f(x - ct) + g(x + ct) \end{aligned}$$

The functions f and g could in fact be any function. Consider the effect of the passage of time on the function f. As t increases the graph of f shifts to the right (at speed c). Similarly g represents a wave travelling to the left.

Using the initial conditions, $y = y_0$ when $t = 0$ gives

$$y_0 = f(x) + g(x) \tag{6}$$

Differentiating to use $\frac{\partial y}{\partial t} = v_0(x)$ gives

$$\begin{aligned} v_0(x) &= -cf'(x) + cg'(x) \\ -cf(x) + cg(x) &= \int_a^x v_0(s)\ \mathrm{d}s \end{aligned} \tag{7}$$

Equations (6) and (7) are a pair of simultaneous equations for f and g. Solving these gives

$$\begin{aligned} f(x) &= \frac{1}{2}y_0(x) - \frac{1}{2c}\int_a^x v_0(s)\ \mathrm{d}s \\ g(x) &= \frac{1}{2}y_0(x) + \frac{1}{2c}\int_a^x v_0(s)\ \mathrm{d}s \end{aligned}$$

The general solution to the 1 dimensional wave equation, equation (5), is therefore

$$y(x,t) = \frac{1}{2}y_0(x+ct) + \frac{1}{2}y_0(x-ct) + \frac{1}{2c}\int_{x-ct}^{x+ct} v_0(s)\ \mathrm{d}s \tag{8}$$

This D'Allembert's solution to the 1 dimensional wave equation.

(24.1.3) First Order Quasi-Linear Partial Differential Equations

A first order quasi-linear partial differential equation is of the form

$$a(u,x,y)\frac{\partial u}{\partial x} + b(u,x,y)\frac{\partial u}{\partial y} = c(u,x,y) \tag{9}$$

where $u = u(x,y)$. The solution can be considered as a surface in xyu space, say $u = F(x,y)$. Let $G(x,y,u) = F(x,y) - u$ so that the normal to the solution surface can be calculated to be

$$\nabla G = \left(\frac{\partial G}{\partial x}, \frac{\partial G}{\partial y}, \frac{\partial G}{\partial u}\right) = \left(\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, -1\right) = \left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, -1\right)$$

Now define $\mathbf{a} = (a,b,c)$ so that equation (9) can be written as $\mathbf{a} \cdot \nabla G = 0$. Hence ∇G is perpendicular to $\mathbf{a}$. But ∇G is normal to the solution surface, therefore $\mathbf{a}$ must lie in the solution surface.

Define now some line, parameterised in terms of τ, say. So

$$(x(\tau), y(\tau), z(\tau)) \quad \text{is parallel to } \mathbf{a}$$

In order to be parallel, the conditions

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = a \qquad \frac{\mathrm{d}y}{\mathrm{d}\tau} = b \qquad \frac{\mathrm{d}u}{\mathrm{d}\tau} = c$$

must hold. This defines a family of lines: as no initial point has been specified the line could be anywhere, but where it is depends on $\mathbf{a}$ which is a function of xyu space. These lines are called characteristics, and the next task is to 'pin them down' so that they define a surface—the solution surface.

The initial conditions are given values of u on some line in the xy plane. Define the corresponding line Γ in xyu space and parameterise it using s, say. So

$$\Gamma(s) = (x(s), y(s), u(s))$$

Define $\tau = 0$ on Γ, together with the rest of the initial conditions. Subject to changing variables back from τ and s to x, y, and u, the equation is now solved. For any point in xyu space, start on the line Γ, then follow a characteristic to the point required.

The projections of the characteristics onto the xy plane are called the characteristic base curves.

In actually using this method the three first order ordinary differential equations may be solved in a number of ways. It is important to remember that the constant of integration can be any function of s. Finding this function is usually quite easy, using the initial conditions

$$x = s \quad u = u_0 \quad \tau = 0 \quad \text{when} \quad y = 0$$

When The Method Of Characteristics Fails

If the characteristic base curves do not cover the entire xy plane, there will be areas of xyu space that cannot be reached, so the solution will not be complete. Often this is due to physical interpretations of the equation being solved, for example that only real values of x, y, and u are allowed.

Definition 10 *An envelope is a curve that bounds the characteristic base curves*

Definition 11 *If ∇u is singular on an envelope then the envelope is called a caustic.*

It can be shown that characteristic base curves end on a caustic. An envelope is just the boundary between the area of the plane in which characteristic base curves lie, and the area in which they do not.

The solution can fail in other ways too. If the line on which the initial conditions are specified is parallel to the characteristic base curves then it will not be possible to use the characteristics to 'move off' the initial line Γ. Essentially, information cannot propagate to other parts of xyu space. This problem usually arises if a mistake has been made translating a physical problem to the partial differential equation to be solved.

Another problem is if the characteristic base curves intersect the line of initial conditions at more than one place. Even if the values at the intersection are consistent (which is unlikely) there is clearly a physical nonsense about such a situation. In such situations the initial conditions (when $y = 0$) may be specified on the half line $x < 0$ say.

Finally, the method may fail if characteristic base curves intersect eachother. This can happen when they are piecewise defined on x for example if

$$y(x) = \begin{cases} s + x & x < 0 \\ s & x > 0 \end{cases}$$

in this case the diagonal lines from $x < 0$ will intersect the vertical lines in the region $x > 0$.

(24.1.4) Solution To Poisson's Equation Using Green's Functions

Poisson's equation is $\nabla^2 u(\mathbf{r}) = \rho(\mathbf{r})$ where ρ is a known function. The solution is sought in some region V that has boundary S. The boundary condition is $u = f(\mathbf{r})$ on S.

Now, where $\mathbf{n}$ is the unit normal vector to the bounding surface S of a volume V; and where $\phi(\mathbf{r})$ and $\psi(\mathbf{r})$ are functions, Green's second theorem gives

$$\int_V \phi\nabla^2\psi - \psi\nabla^2\phi \; \mathrm{d}V = \int_S (\phi\nabla\psi - \psi\nabla\phi) \cdot \mathbf{n} \; \mathrm{d}S$$

As integrals are easier to solve that partial differential equations—they can at least be done numerically—this is used to find an expression for u; the solution to Poisson's equation.

Consider a function $G(\mathbf{r}, \mathbf{r}_0)$ which is a Green's function, so that $\nabla^2 G(\mathbf{r}, \mathbf{r}_0) = \delta(\mathbf{r} - \mathbf{r}_0)$ where δ is the Dirac delta function. Furthermore, define G in such a way that $G = 0$ on S. Now using Green's second theorem;

$$\begin{aligned}
\int_V u(\mathbf{r})\nabla^2 G(\mathbf{r}, \mathbf{r}_0) - G(\mathbf{r}, \mathbf{r}_0)\nabla^2 u(\mathbf{r})\,\mathrm{d}V &= \int_S u(\mathbf{r})\frac{\partial G(\mathbf{r}, \mathbf{r}_0)}{\partial \mathbf{n}} - G(\mathbf{r}, \mathbf{r}_0)\frac{\partial u(\mathbf{r})}{\partial \mathbf{n}}\,\mathrm{d}S \\
\int_V u(\mathbf{r})\delta(\mathbf{r} - \mathbf{r}_0) - G(\mathbf{r}, \mathbf{r}_0)\rho(\mathbf{r})\,\mathrm{d}V &= \int_S f(\mathbf{r})\frac{\partial G(\mathbf{r}, \mathbf{r}_0)}{\partial \mathbf{n}}\,\mathrm{d}S \\
u(\mathbf{r_0}) - \int_V G(\mathbf{r}, \mathbf{r}_0)\rho(\mathbf{r})\,\mathrm{d}V &= \int_S f(\mathbf{r})\frac{\partial G(\mathbf{r}, \mathbf{r}_0)}{\partial \mathbf{n}}\,\mathrm{d}S \\
u(\mathbf{r_0}) &= \int_V G(\mathbf{r}, \mathbf{r}_0)\rho(\mathbf{r})\,\mathrm{d}V + \int_S f(\mathbf{r})\frac{\partial G(\mathbf{r}, \mathbf{r}_0)}{\partial \mathbf{n}}\,\mathrm{d}S
\end{aligned}$$

As $\mathbf{r}_0$ can be chosen at will, this is a solution for Poisson's equation. The problem remaining now is to find the Green's function G.

Let $G(\mathbf{r}, \mathbf{r}_0) = F(\mathbf{r}, \mathbf{r}_0) + H(\mathbf{r})$ where $\nabla^2 F = \delta(\mathbf{r} - \mathbf{r}_0)$, $\nabla^2 H = 0$ and $F + H = 0$ on S. F is called the fundamental solution.

F In Three Dimensions

The task is to solve $\nabla^2 F = \delta(\mathbf{r} - \mathbf{r}_0)$. Notice this problem is spherically symmetric about the point $\mathbf{r}_0$, so consider a sphere S' with radius R and centre $\mathbf{r}_0$. Integrating over S',

$$\int_{V'} \nabla^2 F(\mathbf{r}, \mathbf{r}_0)\,\mathrm{d}V' = \int_{V'} \delta(\mathbf{r} - \mathbf{r}_0)\,\mathrm{d}V' = 1 \quad \text{since } \mathbf{r}_0 \in V' \tag{12}$$

Now using the divergence theorem

$$\int_{V'} \nabla^2 F(\mathbf{r}, \mathbf{r}_0)\,\mathrm{d}V' = \int_{S'} \nabla F(\mathbf{r}, \mathbf{r}_0) \cdot \mathbf{n}\,\mathrm{d}S'$$

but by symmetry, $F(\mathbf{r}, \mathbf{r}_0) = F(r)$ where $r = |\mathbf{r} - \mathbf{r}_0|$. Furthermore, $\mathbf{n}$ always points in the direction $\mathbf{r} - \mathbf{r}_0$. Now, since on S', $r = R$ this gives

$$\begin{aligned}
&= \left[\frac{\mathrm{d}F}{\mathrm{d}r}\right]_{r=R} \int_0^{2\pi} \int_0^{\pi} R^2 \sin\theta\,\mathrm{d}\phi\,\mathrm{d}\theta \\
&= 2\pi R^2 \left[\frac{\mathrm{d}F}{\mathrm{d}r}\right]_{r=R} [-\cos\theta]_0^{\pi} \\
&= 4\pi R^2 \left[\frac{\mathrm{d}F}{\mathrm{d}r}\right]_{r=R}
\end{aligned}$$

Now using equation (12)

$$\begin{aligned}
\frac{\partial F}{\partial r} &= \frac{1}{4\pi r^2} \quad \text{on } r = R \\
F &= \int \frac{1}{4\pi r^2}\,\mathrm{d}r + C \\
&= \frac{-1}{4\pi r} + C
\end{aligned}$$

Imposing the condition $F \to 0$ as $r \to \infty$ gives $C = 0$. Hence

$$F(\mathbf{r}, \mathbf{r}_0) = \frac{-1}{4\pi|\mathbf{r} - \mathbf{r}_0|} \tag{13}$$

F In Two Dimensions

The task is again to solve $\nabla^2 F = \delta(\mathbf{r} - \mathbf{r}_0)$. Integrating over some area A gives

$$\int_A \nabla^2 F(\mathbf{r}, \mathbf{r}_0)\ \mathrm{d}A = \int_A \delta(\mathbf{r} - \mathbf{r}_0)\ \mathrm{d}A = 1$$

Since the problem has circular symmetry about $\mathbf{r}_0$ let A be a circle centred at $\mathbf{r}_0$. Say the circle has boundary B with outward pointing normal vector $\mathbf{n}$ then by the divergence theorem

$$\int_A \nabla^2 F(\mathbf{r}, \mathbf{r}_0)\ \mathrm{d}A = \int_B \nabla F(\mathbf{r}, \mathbf{r}_0) \cdot \mathbf{n}\ \mathrm{d}B = 1$$

Now let $r = |\mathbf{r} - \mathbf{r}_0|$, then since $\mathbf{r} - \mathbf{r}_0$ is parallel to $\mathbf{n}$

$$\begin{aligned}
1 &= \int_B \nabla F(\mathbf{r}, \mathbf{r}_0) \cdot \mathbf{n}\ \mathrm{d}B \\
&= \int_B \frac{\mathrm{d}F(r)}{\mathrm{d}r} \cdot\ \mathrm{d}r \\
&= \left.\frac{\mathrm{d}F(r)}{\mathrm{d}r}\right|_{r=R} \int_0^{2\pi} R^2\ \mathrm{d}\theta \quad \text{by changing to polar co-ordinates} \\
&= 2\pi R\ \left.\frac{\mathrm{d}F(r)}{\mathrm{d}r}\right|_{r=R} \\
\frac{\mathrm{d}F}{\mathrm{d}r} &= \frac{1}{2\pi r} \quad \text{on } r = R \\
F &= \frac{1}{2\pi} \ln r + C \\
&= \frac{1}{2\pi} \ln |\mathbf{r} - \mathbf{r}_0| + C
\end{aligned}$$

Note the constant does not necessarily vanish since $F \nrightarrow 0$as $|\mathbf{r}| \to \infty$, though it may well do in certain applications.

Finding The Green's Function G

Both the 2 and 3 dimensional cases for F have been covered. What now remains is to find the function H such that $\nabla^2 H = 0$ and $F + H = 0$ on S.

H can in fact be constructed from F. Simply let $H = F(\mathbf{r}, \mathbf{r}_1)$ where $\mathbf{r}_1 \notin V$, the solution region. This means that $\nabla^2 H = 0$ in V and $F + H = 0$ on S.

A typical choice for $\mathbf{r}_1$ is the reflection of $\mathbf{r}_0$ in the bounding line or surface of V.

Poisson's Formula

When solving $\nabla^2 u = 0$ in 2 dimensions on the upper half plane the above theory yields a relatively easy result. Proceeding with the method of Green's functions, since $\rho = 0$ the result reduces to

$$u(x_0, y_0) = \int_S f \frac{\partial G}{\partial \mathbf{n}}\ \mathrm{d}S$$

In 2 dimensions

$$G(\mathbf{r}, \mathbf{r}_0) = \frac{1}{2\pi} \ln |\mathbf{r} - \mathbf{r}_0| + C$$

and $\nabla^2 G = \delta(\mathbf{r} - \mathbf{r}_0)$. The task now is to modify G to make it zero on the bounding 'surface', the x axis. For $\mathbf{r}_1$ in the lower half plane let

$$G = \frac{1}{2\pi} \ln |\mathbf{r} - \mathbf{r}_0| - \frac{1}{2\pi} \ln |\mathbf{r} - \mathbf{r}_1| + C$$

then by symmetry $\nabla^2 G = \delta(\mathbf{r} - \mathbf{r}_0) - \delta(\mathbf{r} - \mathbf{r}_1) = \delta(\mathbf{r} - \mathbf{r}_0)$ since $\mathbf{r}_1$ not in the upper half plane, so this condition still holds. Say $\mathbf{r}_0 = x\mathbf{i} + y\mathbf{j}$ so let $\mathbf{r}_1 = x\mathbf{i} - y\mathbf{j}$. Hence

$$G(\mathbf{r}, \mathbf{r}_0) = \frac{1}{2\pi} \ln \sqrt{(x - x_0)^2 + (y - y_0)^2} - \frac{1}{2\pi} \ln \sqrt{(x - x_0)^2 + (y + y_0)^2} + C$$

On the boundary $y = 0$, and substituting this in gives $G = 0 \Leftrightarrow C = 0$. Let $\mathbf{n}$ be the outward pointing normal vector to the boundary, so $\mathbf{n} = -\mathbf{j}$. Hence

$$\begin{aligned}
\frac{\partial G}{\partial \mathbf{n}} &= -\nabla G \cdot \mathbf{j} = -\frac{\partial G}{\partial y} \\
&= -\frac{1}{2\pi} \frac{1}{2} \frac{2(y - y_0)}{(x - x_0)^2 + (y - y_0)^2} + \frac{1}{2\pi} \frac{1}{2} \frac{2(y + y_0)}{(x - x_0)^2 + (y + y_0)^2} \\
\left.\frac{\partial G}{\partial \mathbf{n}}\right|_{y=0} &= -\frac{1}{2\pi} \frac{-y_0}{(x - x_0)^2 + y_0^2} + \frac{1}{2\pi} \frac{y_0}{(x - x_0)^2 + y_0^2} \\
&= \frac{y_0}{\pi \left((x - x_0)^2 + y_0^2\right)}
\end{aligned}$$

The final result is Poisson's formula for the upper half plane. The solution to $\nabla^2 u = 0$ is given by

$$u(x, y) = \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{f(s)}{(s - x)^2 + y^2} \, \mathrm{d}s$$

Particular choices of f (in particular constants and zero) will allow this integral to be calculated quite easily. It should be remembered that

$$\int \frac{1}{x^2 + a^2} \, \mathrm{d}x = \frac{1}{a} \arctan\left(\frac{x}{a}\right) \quad \text{and} \quad \lim_{x \to \pm\infty} \arctan x = \frac{\pm\pi}{2}$$

(24.1.5) Conformal Mappings

Results from Complex Variable Theory (Chapter **??**) are assumed. The idea of a conformal mapping is to change co-ordinate systems in such a way that a partial differential equation becomes easier to solve. The solution can then be transformed back to the original co-ordinate system.

Definition 14 *A conformal mapping is a function $f : \mathbb{C} \to \mathbb{C}$ that preserves angles.*

It can be shown that any analytic function is a conformal mapping provided its derivative is never zero.

It is common to treat $\mathbb{R}^2$ as $\mathbb{C}$ and map to a different complex plane in which the area under consideration—the area in which the partial differential equation must be solved—maps to the upper half plane. Once in the new co-ordinate system boundary conditions must also be converted, allowing a solution to be found. A number of common transformations are now presented for memorisation.

In each case a mapping $z = x + iy$ to $w = u + iv$ is sought.

1. Let D be the wedge $0 \leqslant \arg z \leqslant \frac{\pi}{m}$ for $m \geqslant \frac{1}{2}$ and consider the mapping $w = z^m$.

   Since $z_1 z_2$ has argument $\arg z_1 + \arg z_2$ the half line $\frac{\pi}{m}$ is mapped to the negative u axis. The positive x axis is clearly mapped to the positive u axis.

Clearly D is mapped to the upper half plane with its boundaries mapping to the positive and negative parts of the u axis.

2. Let D be the strip $0 \leqslant y \leqslant a$ and consider the mapping $w = e^{\frac{\pi z}{a}}$. Clearly the derivative is positive.

$$w = e^{\frac{\pi z}{a}} = e^{\frac{\pi}{a}(x+iy)} = e^{\frac{\pi x}{a}} e^{\frac{i\pi y}{a}}$$

If $y = 0$ then $w = e^{\frac{\pi x}{a}} \in \mathbb{R}^+$ hence the lower boundary of the strip is mapped to the positive u axis.
If $y = a$ then $w = e^{\frac{\pi x}{a}} e^{i\pi} = -e^{\frac{\pi x}{a}} \in \mathbb{R}^-$ hence the upper boundary of the strip is mapped to the negative u axis.
If $0 < y < a$ then since $w = e^{\frac{\pi x}{a}} \left(\cos\left(\frac{\pi y}{a}\right) + i \sin\left(\frac{\pi y}{a}\right)\right)$ clearly the imaginary part is positive, so D is mapped to the upper half plane.

3. Let D be a semi-infinite strip of width a such that $\frac{-a}{2} \leqslant x \leqslant \frac{a}{2}$ and $y > 0$ and consider the mapping $w = \sin\left(\frac{\pi z}{a}\right)$.

$$\begin{aligned} w &= \sin\left(\frac{\pi}{a}(x+iy)\right) \\ &= \sin\left(\frac{\pi x}{a}\right)\cos\left(\frac{i\pi y}{a}\right) + \cos\left(\frac{\pi x}{a}\right)\sin\left(\frac{i\pi y}{a}\right) \\ &= \sin\left(\frac{\pi x}{a}\right)\cosh\left(\frac{\pi y}{a}\right) + i\cos\left(\frac{\pi x}{a}\right)\sinh\left(\frac{\pi y}{a}\right) \end{aligned}$$

because

$$\sin(i\theta) = \frac{e^{i^2\theta} - e^{-i^2\theta}}{2i} = -i\frac{e^{-\theta} - e^{\theta}}{2} = i \sinh\theta$$

and similarly

$$\cos(i\theta) = \frac{e^{i^2\theta} + e^{-i^2\theta}}{2} = \frac{e^{-\theta} + e^{\theta}}{2} = \cosh\theta$$

Now, if $x = \frac{a}{2}$ then $w = \cosh\frac{\pi y}{a} \in \mathbb{R}^+$. But $\cosh\theta \geqslant 1$ for all θ, so the left boundary of the strip is mapped to the u axis for $u \geqslant 1$.
If $x = \frac{-a}{2}$ then $w = -\cosh\frac{\pi y}{a}$ which by the same argument as above is the u axis for $u \leqslant -1$.
If $\frac{-a}{2} < x < \frac{a}{2}$ then $-1 < w < 1$ which accounts for the remaining part of the u axis. Finally, observe that $\frac{-a}{2} \leqslant x \leqslant \frac{a}{2}$ gives $v \geqslant 0$ so this region is indeed the upper half plane.

4. The Möbius transformation can be used to map polygons and circles to the upper half plane. Let $w = \frac{az+b}{cz+d}$ with $ad - bd \neq 0$. In particular the transformation to use for the unit circle is $w = \left|\frac{z-i}{z+i}\right|$.

Theorem 15 (Riemann's Mapping Theorem) *Let D and D' be dimply connected domains. Then there exists a function $w(z)$ which is analytic in D which conformally maps D to D'.*

Riemann's Mapping Theorem shows that a conformal mapping can always be found, but does not suggest how. The four mappings presented above should suffice for most purposes.

The method for solving a partial differential equation by use of conformal mappings is as follows.

1. Choose the conformal mapping and convert the boundary conditions.
2. Solve the differential equation in (u, v).
3. In the solution, substitute for (u, v) with (x, y) using the equations determined by the chosen mapping.

Example 16 *Solve $\nabla^2 T = 0$ in the infinite wedge of angle $\frac{\pi}{3}$. The boundary $\theta = 0$ is maintained at temperature T_0 while the boundary $\theta = \frac{\pi}{3}$ is maintained at temperature T_1.*

Proof. Solution Using the conformal mapping $z \mapsto z^3$ the problem is transformed to

$$\frac{\partial^2 T}{\partial u^2} + \frac{\partial^2 T}{\partial v^2} = 0 \quad \text{with} \quad (u,0) = \begin{cases} T_0 & u > 0 \\ T_1 & u < 0 \end{cases}$$

Using Poisson's formula,

$$\begin{aligned}
T(u,v) &= \frac{v}{\pi} \int_{-\infty}^{\infty} \frac{f(s)}{(u-s)^2 + v^2}\, \mathrm{d}s \\
&= \frac{vT_0}{\pi} \int_0^{\infty} \frac{1}{(u-s)^2 + v^2}\, \mathrm{d}s + \frac{vT_1}{\pi} \int_{-\infty}^{0} \frac{1}{(u-s)^2 + v^2}\, \mathrm{d}s \\
&= \frac{T_0}{\pi} \left[\arctan\left(\frac{s-u}{v} \right) \right]_0^{\infty} + \frac{T_1}{\pi} \left[\arctan\left(\frac{s-u}{v} \right) \right]_{-\infty}^{0} \\
&= \frac{T_0}{2} - \frac{T_0}{\pi} \arctan\left(\frac{-u}{v} \right) + \frac{T_1}{\pi} \arctan\left(\frac{-u}{v} \right) + \frac{T_1}{2} \\
&= \frac{T_0 + T_1}{2} + \frac{T_1 - T_0}{\pi} \arctan\left(\frac{-u}{v} \right)
\end{aligned}$$

Now, using DeMoivre's theorem

$$w = u + iv = (x+iy)^3 = (r\cos\theta + ri\sin\theta)^3 = r^3 \cos 3\theta + ir^3 \sin 3\theta$$

So $u = \cos 3\theta$ and $v = \sin 3\theta$ hence

$$T(r,\theta) = \frac{T_0 + T_1}{2} + \frac{T_1 - T_0}{\pi} \arctan\left(-\cot 3\theta\right)$$ □

(24.1.6) Well Posed Problems

It is not necessarily the case that any partial differential equation will have a solution. Indeed, even if a solution exists it may break now in part of the solution region.

Definition 17 *A boundary value problem is well posed if the following criteria are met.*

1. *Existence—the problem has a solution.*
2. *Uniqueness—the problem has only one solution.*
3. *Continuous dependence on boundary values—a small change in the boundary values causes a small change in the solution.*

The first two criteria are rather obvious, the third requires a little more attention. It essentially means that the solution is quite stable—if the boundary temperatures in a diffusion equation problem were altered slightly, it would be reasonable to expect to have a similar solution. The usual way to show the third criterion is to consider a perturbation.

The 1 Dimensional Wave Equation

D'allembert's solution is

$$y(x,t) = \frac{1}{2}\left(y_0(x+ct) + y_0(x-ct)\right) + \frac{1}{2c} \int_{x-ct}^{x+ct} v_0(s)\, \mathrm{d}s$$

Consider two solutions $y_1(x)$ and $y_2(x)$ with initial conditions

$$y_1(x,0) = y_0(x) \qquad \frac{\partial y_1}{\partial t} = v_0(x) \qquad y_2(x,0) = y_0(x) + \overline{y}(x) \qquad \frac{\partial y_2}{\partial t} = v_0(x) + \overline{v}(x)$$

The difference between the two solutions is then

$$\begin{aligned} Y = y_2(x,t) - y_1(x,t) &= \left(\frac{1}{2}\left(y_0(x+ct) + \overline{y}(x+ct) + y_0(x-ct) + \overline{y}(x-ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} v_0(s) + \overline{v}(s)\,\mathrm{d}s\right) - \left(\frac{1}{2}\left(y_0(x+ct) + y_0(x - \right.\right. \\ &= \frac{1}{2}\left(\overline{y}(x+ct) + \overline{y}(x-ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} \overline{v}(s)\,\mathrm{d}s \end{aligned}$$

Hence $|Y| \leqslant \max|\overline{y}| + t\max|\overline{v}|$. Hence

$$\forall \varepsilon > 0 \quad \exists \delta > 0 \quad \text{such that} \quad \max\left(\max|\overline{y}|, \max|\overline{v}|\right) < \delta \Rightarrow |Y| < \varepsilon$$

Hence criteria 3 holds, and the problem is well posed.

Initial Value Problem For Laplace's Equation

Consider Laplace's equation on the upper half plane with $u(x,0) = u_0(x)$ and $\left.\frac{\partial u}{\partial y}\right|_{y=0} = v_0(x)$. Seeking a particular solution, in the special case $u_0 = 0$ and $v_0 = \frac{1}{\lambda}\sin\lambda x$. A solution is

$$\hat{u}(x,y) = \frac{1}{\lambda^2}\sin\lambda x \sinh\lambda y$$

As $\lambda \to \infty$ clearly $v_0 \to 0$ but for $y \neq 0$

$$\max|\hat{u}| = \frac{\sinh\lambda y}{\lambda^2} = \frac{e^{\lambda y} - e^{-\lambda y}}{2\lambda^2} \to \infty$$

Consider now the general problem as initially stated. Since $\hat{u}$ is a particular solution, it can form part of a general solution $u = U + \hat{u}$. As $\lambda \to \infty$ the boundary conditions are still satisfied but $\hat{u} \to \infty$ and hence $u \to \infty$. This shows the problem to be ill posed since

$$\forall \varepsilon > 0 \quad \exists K > o \quad \text{such that} \quad \max\left|\left.\frac{\partial u}{\partial y}\right|_{y=0}\right| < \varepsilon \quad \text{and} \quad \max|u(x,y)| > K$$

(24.1.7) Second Order Partial Differential Equations

Homogeneous Equations & Classification

A second order homogeneous partial differential equation may always be expressed in the form

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} = 0$$

where A, B, and C are constants. In this simple case it is quite easy to find a solution. Take some function $f(x + \lambda y)$ then

$$\frac{\partial u}{\partial x} = f'(x+\lambda y) \quad \frac{\partial^2 u}{\partial x^2} = f''(x+\lambda y) \quad \frac{\partial u}{\partial y} = \lambda f'(x+\lambda y) \quad \frac{\partial^2 u}{\partial y^2} = \lambda^2 f''(x+\lambda y) \quad \frac{\partial^2 u}{\partial x \partial y} = \lambda f''(x+\lambda y)$$

Substituting this into the differential equation gives

$$\begin{aligned}
Af''(x+\lambda y) + B\lambda f''(x+\lambda y) + C\lambda^2 f''(x+\lambda y) &= 0 \\
f''(x+\lambda y)\left(A + B\lambda + C\lambda^2\right) &= 0 \\
\text{so} \quad f''(x+\lambda y) &= 0 \\
\text{or} \quad A + B\lambda + C\lambda^2 &= 0 \\
\text{i.e.} \quad \lambda &= \frac{-B \pm \sqrt{B^2 - 4AC}}{2C}
\end{aligned}$$

The nature of λ is used to classify the equation.

- If λ has two distinct values, so $B^2 - 4AC > 0$, then the equation is hyperbolic.
- If λ is repeated, so $B^2 - 4AC = 0$, then the equation is parabolic.
- If λ is complex, so $B^2 - 4AC < 0$, then the equation is elliptic.

In the case of a hyperbolic equation the solution will be of the form $u = f(x + \lambda_1 y) + g(x + \lambda_2 y)$, and similarly for the elliptic case. However, in the parabolic case part of the solution is 'missing'. It is readily shown (by differentiating and substituting) that $u = h(x,y)g\left(x + \frac{B}{2C}\right)$ where h turns out to be $h(x,y) = x$. The proper solution is then

$$u(x,y) = f\left(x - \frac{B}{2C}\right) + xg\left(x - \frac{B}{2C}\right)$$

General Second Order Partial Differential Equations

A general linear second order partial differential equation has the form

$$A(x,y)\frac{\partial^2 u}{\partial x^2} + B(x,y)\frac{\partial^2 u}{\partial x \partial y} + C(x,y)\frac{\partial u}{\partial y} = F\left(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right) \tag{18}$$

where F may in fact be non-linear. One of three possible types of boundary condition may apply.

- Dirichlet: u is specified on the boundary.
- Neumann: $\frac{\partial u}{\partial \mathbf{n}} = \nabla u \cdot \mathbf{n}$ is specified on the boundary, where $\mathbf{n}$ is the normal unit vector.
- Cauchy: u and $\frac{\partial u}{\partial \mathbf{n}}$ are specified on a curve in the solution region.

Consider the Cauchy problem for a curve C with normal vector $\mathbf{n}$ and tangent vector $\mathbf{r}$. Let s be the arc length along C and say

$$u = \phi(s) \quad \frac{\partial u}{\partial \mathbf{n}} = \psi(s)$$

on C. At any point on C

$$\mathrm{d}\mathbf{r} = \mathrm{d}x\mathbf{i} + \mathrm{d}y\mathbf{j} \quad \text{and} \quad \mathbf{n}\,\mathrm{d}s = \mathrm{d}y\mathbf{i} - \mathrm{d}x\mathbf{j}$$

Hence

$$\frac{\partial u}{\partial s} = \nabla u \cdot \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}s} = \frac{\partial u}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}s} + \frac{\partial u}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}s} = \frac{\mathrm{d}\phi}{\mathrm{d}s} \tag{19}$$

$$\frac{\partial u}{\partial \mathbf{n}} = \nabla u \cdot \mathbf{n} = \frac{\partial u}{\partial x}\frac{\mathrm{d}y}{\mathrm{d}s} - \frac{\partial u}{\partial y}\frac{\mathrm{d}x}{\mathrm{d}s} = \psi \tag{20}$$

These are simultaneous equations for $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ on C. Using the chain rule

$$\frac{\mathrm{d}}{\mathrm{d}s} = \frac{\mathrm{d}x}{\mathrm{d}s}\frac{\mathrm{d}}{\mathrm{d}x} + \frac{\mathrm{d}y}{\mathrm{d}s}\frac{\mathrm{d}}{\mathrm{d}y}$$

Hence

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\partial u}{\partial x}\right) = \frac{\mathrm{d}x}{\mathrm{d}s}\frac{\partial^2 u}{\partial x^2} + \frac{\mathrm{d}y}{\mathrm{d}s}\frac{\partial^2 u}{\partial x \partial y} \tag{21}$$

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\partial u}{\partial y}\right) = \frac{\mathrm{d}x}{\mathrm{d}s}\frac{\partial^2 u}{\partial x \partial y} + \frac{\mathrm{d}y}{\mathrm{d}s}\frac{\partial^2 u}{\partial y^2} \tag{22}$$

Equations (18), (21), and (22) are simultaneous equations for the second partial derivatives of u. There is no solution when

$$\begin{vmatrix} A & B & C \\ \frac{\mathrm{d}x}{\mathrm{d}s} & \frac{\mathrm{d}y}{\mathrm{d}s} & 0 \\ 0 & \frac{\mathrm{d}x}{\mathrm{d}s} & \frac{\mathrm{d}y}{\mathrm{d}s} \end{vmatrix} = 0$$

$$\Leftrightarrow A\left(\frac{\mathrm{d}y}{\mathrm{d}s}\right)^2 - B\frac{\mathrm{d}x}{\mathrm{d}s}\frac{\mathrm{d}y}{\mathrm{d}s} + C\left(\frac{\mathrm{d}x}{\mathrm{d}s}\right)^2 = 0$$

now multiply through by $\left(\frac{\mathrm{d}s}{\mathrm{d}x}\right)^2$ to give

$$A\left(\frac{\mathrm{d}y}{\mathrm{d}s}\frac{\mathrm{d}s}{\mathrm{d}x}\right)^2 - B\left(\frac{\mathrm{d}x}{\mathrm{d}s}\frac{\mathrm{d}s}{\mathrm{d}x}\frac{\mathrm{d}y}{\mathrm{d}s}\frac{\mathrm{d}s}{\mathrm{d}x}\right) + C\left(\frac{\mathrm{d}x}{\mathrm{d}s}\frac{\mathrm{d}s}{\mathrm{d}x}\right)^2 = 0$$

$$A\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 - B\frac{\mathrm{d}y}{\mathrm{d}x} + C = 0 \tag{23}$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{B \pm \sqrt{B^2 - 4AC}}{2A} \tag{24}$$

Equation (23) defines the curves in the xy plane on which the second partial derivatives of u cannot be found. These curves are called characteristics and can be determined using equation (24).

- When $B^2 - 4AC > 0$ (hyperbolic) two families of curves in the xy plane are produce.
- When $B^2 - 4AC = 0$ (parabolic) one family of curves in the xy plane is produced.
- When $B^2 - 4AC < 0$ (elliptic) two families of curves in the xy argand 4-space are produced.

Since A, B, and C are not constant the classification of the equation is local. However, when they are constant the characteristics are simply straight lines.

Along characteristics propagates partial information about the solution. The differential equation can only be solved in regions where the characteristics intersect. It may therefore be impossible to solve the equation after some time T, say. This is illustrated in Figure 24.1.7

Canonical Forms

A simple change of variable in a partial differential equation can drastically simplify its expression. Indeed, the new variables $x + \lambda y$ (for both values of λ) would simplify the solutions to equations. This is in fact precisely the transformation required. Once the equations are simplified, they are said to be in canonical form.

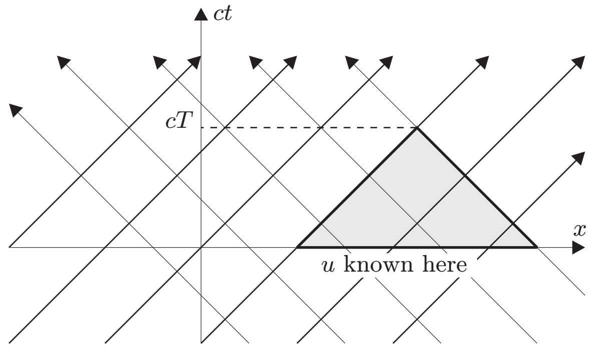

Figure 1: A solution cannot be found after time T.

Example 25 *Consider the equation*

$$4\frac{\partial^2 u}{\partial x^2}+5\frac{\partial^2 u}{\partial x\partial y}+\frac{\partial^2 u}{\partial y^2}=0$$

then the characteristics solve

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{5\pm\sqrt{5^2-4.4.1}}{2.4}=1 \textit{ or } \frac{1}{4}$$

Therefore $y=x+c$ or $y=\frac{1}{4}x+c$ where c is a constant. Hence define the new variables

$$\eta=y-x \qquad \zeta=y-\frac{1}{4}x$$

Hence

$$\begin{aligned}\frac{\partial}{\partial x}&=\frac{\partial\eta}{\partial x}\frac{\partial}{\partial\eta}+\frac{\partial\zeta}{\partial x}\frac{\partial}{\partial\zeta}=-\frac{\partial}{\partial\eta}-\frac{1}{4}\frac{\partial}{\partial\zeta}\\ \frac{\partial}{\partial y}&=\frac{\partial\zeta}{\partial y}\frac{\partial}{\partial\eta}+\frac{\partial\zeta}{\partial y}\frac{\partial}{\partial\zeta}=\frac{\partial}{\partial\eta}+\frac{\partial}{\partial\zeta}\end{aligned}$$

Now substituting into the original equation,

$$\begin{aligned}4\left(-\frac{\partial}{\partial\eta}-\frac{1}{4}\frac{\partial}{\partial\zeta}\right)\left(-\frac{\partial u}{\partial\eta}-\frac{1}{4}\frac{\partial u}{\partial\zeta}\right)+5\left(-\frac{\partial}{\partial\eta}-\frac{1}{4}\frac{\partial}{\partial\zeta}\right)\left(\frac{\partial u}{\partial\eta}+\frac{\partial u}{\partial\zeta}\right)+\left(\frac{\partial}{\partial\eta}+\frac{\partial}{\partial\zeta}\right)\left(\frac{\partial u}{\partial\eta}+\frac{\partial u}{\partial\zeta}\right)&=0\\ 4\frac{\partial^2 u}{\partial\eta^2}+2\frac{\partial^2 u}{\partial\eta\partial\zeta}+\frac{1}{4}\frac{\partial u}{\partial\zeta}-5\frac{\partial^2 u}{\partial\eta^2}-\frac{5}{4}\frac{\partial^2 u}{\partial\eta\partial\zeta}-\frac{\partial^2 u}{\partial\eta\partial\zeta}-\frac{5}{4}\frac{\partial^2 u}{\partial\zeta^2}+\frac{\partial^2 u}{\partial\eta^2}+\frac{\partial^2 u}{\partial\zeta^2}+2\frac{\partial^2 u}{\partial\eta\partial\zeta}&=0\\ \frac{\partial^2 u}{\partial\eta\partial\zeta}&=0\end{aligned}$$

This is in fact a general result for second order hyperbolic partial differential equations with constant coefficients.

More generally the non-homogeneous equation would reduce to

$$\frac{\partial^2 u}{\partial\eta\partial\zeta}=F\left(\eta,\zeta,u,\frac{\partial u}{\partial\eta},\frac{\partial u}{\partial\zeta}\right)$$

Elliptic equations in canonical form are Poisson's equation, and parabolic equations reduce to the form

$$\frac{\partial^2 u}{\partial\zeta^2}=F\left(\eta,\zeta,u,\frac{\partial u}{\partial\eta},\frac{\partial u}{\partial\zeta}\right)$$

which can be solved by integration. It is clear that transforming to canonical form allows the equation to be solved using a method already covered.

In the three cases of hyperbolic, elliptic, and parabolic equations the change of variable chosen is slightly different.

- For hyperbolic equations the characteristic equation is of the form $y = px + c$ where p takes one of two values, p_1 and p_2 say. The change of variables is then $\zeta = y - p_1 x$ and $\eta = y - p_2 x$.
- For elliptic equations there is only one characteristic equation, $y = px + c$ say. The change of variables is $\zeta = y - px$ and $\eta = x$.
- For a parabolic equation there are two characteristic equations but x has a complex coefficient, say $y = (p \pm iq)x + c$. In this case the change of variables is of the form $c = \zeta + i\eta$ and so $\zeta = y - px$ and $\eta = -qx$.

(24.1.8) Integral Transform Methods

Laplace Transforms

Recall that the Laplace transform of a function $g(t)$ is given by

$$G(s) = \mathcal{L}\left(g(t)\right) = \int_0^\infty e^{-st} g(t)\ \mathrm{d}t$$

the function $g(t)$ must be of exponential order. Some results to remember include

$$\begin{aligned}
\mathcal{L}\left(g'(t)\right) &= s\mathcal{L}\left(g(t)\right) - g(0) \\
\mathcal{L}\left(g''(t)\right) &= s^2\mathcal{L}\left(g(t)\right) - sg(0) - g'(0) \\
g * h &= \int_0^t g(y)h(t-y)\ \mathrm{d}y \\
\mathcal{L}\left(g * h\right) &= \mathcal{L}\left(g\right)\mathcal{L}\left(h\right)
\end{aligned}$$

Once a solution has been found under the transform, the inverse transform must be applied. The Bromvich inversion formula for a Laplace transform is

$$g(t) = \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} s^{st} G(s)\ \mathrm{d}s$$

This integral is along an infinite vertical contour in the complex plane of s. The value of γ is chosen such that the contour lies to the right of any singularities. Such an integral can be evaluated by considering a semicircle and using methods involving complex residues

Example 26 *A simple example is the one dimensional wave equation.*

$$\frac{\partial^2 y}{\partial t^2} = \frac{1}{c^2}\frac{\partial^2 y}{\partial x^2} \quad \textit{with} \quad y(x,0) = 0 \quad \left.\frac{\partial y}{\partial t}\right|_{t=0} = 0 \quad y(0,t) = g(t)$$

Taking Laplace transforms with respect to t gives

$$\begin{aligned}
\mathcal{L}\left(\frac{\partial^2 y}{\partial x^2}\right) &= \frac{\partial^2 \overline{y}}{\partial x^2} \\
\mathcal{L}\left(\frac{\partial^2 y}{\partial t^2}\right) &= s^2\overline{y} - sy(x,0) - \left.\frac{\partial y}{\partial t}\right|_{t=0} \\
&= s^2\overline{y} \\
\mathcal{L}\left(g(t)\right) &= G(s)
\end{aligned}$$

Constructing the Laplace transform of the differential equation gives

$$\frac{\partial^2 \overline{y}}{\partial x^2} - \frac{s^2}{c^2}\overline{y} = 0$$

Treating this as an ordinary differential equation in x gives

$$\overline{y} = A(s)e^{\frac{-sx}{c}} + B(s)e^{\frac{sx}{c}}$$

Imposing the condition $\overline{y} \to 0$ as $x \to \infty$ gives $B(s) = 0$. Putting $x = 0$ gives $A(s) = G(s)$. Now using the Bromvich inversion formula

$$\begin{aligned}
y &= \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} G(s)e^{\frac{-sx}{c}}e^{st}\,\mathrm{d}s \\
&= \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} G(s)e^{s\left(t-\frac{x}{c}\right)}\,\mathrm{d}s \\
\textit{but } g(t) &= \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} G(s)e^{st}\,\mathrm{d}s \\
\textit{hence } y &= g\left(t - \frac{x}{c}\right)
\end{aligned}$$

A similar process can be performed for the diffusion equation

$$\frac{\partial T}{\partial t} = D\frac{\partial^2 T}{\partial x^2} \quad T(x,0) = 0 \quad T(0,t) = \sin(\omega t) \quad x \geqslant 0$$

Taking Laplace transforms with respect to t gives $D\frac{\partial^2 \overline{T}}{\partial x^2} - s\overline{T} = 0$. Recall that $\mathcal{L}(\sin(\omega t)) = \frac{\omega}{\omega^2 + s^2}$ and hence

$$\begin{aligned}
\overline{T} &= \frac{\omega}{\omega^2 + s^2}\exp\left(-x\sqrt{\left(-\frac{s}{D}\right)}\right) \\
\text{so } T &= \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty}\frac{\omega}{\omega^2 + s^2}\exp\left(-x\sqrt{\left(st - \frac{s}{D}\right)}\right)\,\mathrm{d}s \qquad (27)
\end{aligned}$$

There is a problem in that the integrand is not single valued inside the "large D shape" round which it will be integrated since for example where $s = e^{i\theta}$

- At $\theta = 0$ $\sqrt{s} = e^0 = 1$.
- At $\theta = 2\pi$ $\sqrt{s} = e^{i\pi} = -1$.

These problems occur every time θ goes round by another revolution. To stop this a branch cut is made—the negative real axis is removed. The (closed) contour of integration now becomes

1. Quarter circle from $\gamma + iR$ to $-R$*.

*The point $-R$ has been removed by the branch cut, but it is possible to get arbitrarily close to it.

2. Line in the upper half plane from $-R$ to $-\varepsilon$.
3. Circle, clockwise, or radius ε from $-\varepsilon$ to $-\varepsilon$. The limit is taken as $\varepsilon \to 0$.
4. Line in the lower half plane from $-\varepsilon$ to $-R$.
5. Quarter circle from $-R$ to $\gamma - iR$.
6. Straight line from $\gamma - iR$ to $\gamma + iR$.

The value of this integral is $2\pi i$ times the sum of the residues in the interior of the contour. Using this and taking the limit as $R \to \infty$ the Bromvich integral can be evaluated.

Lemma 28 (Jordan's Lemma) *Let* $\gamma_R = \left\{ z \mid z = Re^{i\theta} \text{ for } 0 \leqslant \theta \leqslant \pi \right\}$ *and suppose that* $M(R) = \sup_{z \in \gamma_R} |f(z)|$. *Then if* $\lim_{R \to \infty} M(R) = 0$,

$$\lim_{R \to \infty} \int_{\gamma_R} e^{i\alpha z} f(z) \, \mathrm{d}z = 0$$

for all $\alpha > 0$.

Theorem 29 (The Residue Theorem) *Suppose that* γ *is a simple closed contour in a domain* D*, and let* f *be a complex function which is analytic on* D *except at finitely many points,* $p_1, p_2, \ldots, p_k$*, all of which line in* $\operatorname{Int} \gamma$. *Then*

$$\int_\gamma f(z) \, \mathrm{d}z = 2\pi i \sum_{i=1}^{k} \operatorname{Res}(f, p_i)$$

Returning to the example of the diffusion equation, on the circular parts put $s = Re^{i\theta}$ and hence by Jordan's Lemma these integrals are zero.

On the small circle round the origin put $s = \varepsilon e^{i\theta}$ so $\frac{\mathrm{d}s}{\mathrm{d}\theta} = i\varepsilon e^{i\theta}$. Substituting into the integrand gives

$$\frac{1}{2\pi i} \int_\pi^{-\pi} \frac{\omega}{\omega^2 + \varepsilon^2 e^{2i\theta}} \exp\left(\varepsilon e^{i\theta} - \sqrt{\frac{\varepsilon}{D}} x e^{i\frac{\theta}{2}} \right) \varepsilon i e^{i\theta} \, \mathrm{d}\theta$$

By using series expansions, this integral is asymptotic to

$$\frac{\varepsilon}{2\pi i} \int_\pi^{-\pi} e^{i\theta} \, \mathrm{d}\theta \to 0$$

Hence there is no contribution from the small circle. Finally the two straight line parts along the real axis must be integrated over. For the upper part let $s = -\sigma = \sigma e^{i\pi}$ so that $\frac{\mathrm{d}s}{\mathrm{d}\sigma} = -\sigma$. Remembering to change the limits, this gives

$$\int_\infty^0 \frac{\omega}{\omega^2 + \sigma^2} \exp\left(-\sigma t - x\sqrt{\frac{\sigma e^{i\pi}}{D}} \right) (-1) \, \mathrm{d}\sigma = \int_0^\infty \frac{\omega}{\omega^2 + \sigma^2} \exp\left(-\sigma t - ix\sqrt{\frac{\sigma}{D}} \right) \, \mathrm{d}\sigma$$

Similarly, for the lower part let $s = -\sigma = \sigma e^{-i\pi}$, giving

$$\int_\infty^0 \frac{\omega}{\omega^2 + \sigma^2} \exp\left(-\sigma t - x\sqrt{\frac{\sigma e^{-i\pi}}{D}} \right) (-1) \, \mathrm{d}\sigma = -\int_0^\infty \frac{\omega}{\omega^2 + \sigma^2} \exp\left(-\sigma t + ix\sqrt{\frac{\sigma}{D}} \right) \, \mathrm{d}\sigma$$

Observe these integrals are over the same interval and the exponential terms are complex conjugate. This suggests that when added they will combine to a sine or cosine function. Doing this gives

$$-2i \int_0^\infty \frac{\omega}{\omega^2 + \sigma^2} e^{-\sigma t} \sin\left(x\sqrt{\frac{\sigma}{D}} \right) \, \mathrm{d}\sigma \tag{30}$$

Now, inside this contour there are two singularities—at $s = \pm i\omega$. These are both simple poles[†] and so are easily evaluated.

$$\begin{aligned}
\operatorname{Res}\left(\frac{\omega}{(s+i\omega)(s-i\omega)}\exp\left(st - x\sqrt{\frac{s}{D}}\right), i\omega\right) &= \frac{\omega}{2i\omega}\exp\left(i\omega t - x\sqrt{\frac{i\omega}{D}}\right)\\
\operatorname{Res}\left(\frac{\omega}{(s+i\omega)(s-i\omega)}\exp\left(st - x\sqrt{\frac{s}{D}}\right), -i\omega\right) &= \frac{-\omega}{2i\omega}\exp\left(-i\omega t - x\sqrt{\frac{-i\omega}{D}}\right)
\end{aligned}$$

Now, $\sqrt{i} = \sqrt{e^{i\frac{\pi}{2}}} = e^{i\frac{\pi}{4}} = \frac{1}{\sqrt{2}}(1+i)$. Hence adding these residues and multiplying by $2\pi i$ gives

$$\pi\exp\left(i\omega t - x(1+i)\sqrt{\frac{\omega}{2D}}\right) - \pi\exp\left(-i\omega t - x(1-i)\sqrt{\frac{\omega}{2D}}\right) = 2\pi i\sin\left(\omega t - x\sqrt{\frac{\omega}{2D}}\right)\exp\left(-x\sqrt{\frac{\omega}{2D}}\right)$$

Now using this with equations (27) and (30) and the residue theorem gives

$$\begin{aligned}
2\pi i T &= 2\pi i\sin\left(\omega t - x\sqrt{\frac{\omega}{2D}}\right)\exp\left(-x\sqrt{\frac{\omega}{2D}}\right) + 2i\int_0^\infty \frac{\omega}{\omega^2+\sigma^2}e^{-\sigma t}\sin\left(x\sqrt{\frac{\sigma}{D}}\right)\\
T &= \sin\left(\omega t - x\sqrt{\frac{\omega}{2D}}\right)\exp\left(-x\sqrt{\frac{\omega}{2D}}\right) + \frac{1}{\pi}\int_0^\infty \frac{\omega}{\omega^2+\sigma^2}e^{-\sigma t}\sin\left(x\sqrt{\frac{\sigma}{D}}\right)
\end{aligned}$$

Fourier Transforms

In a similar way to Laplace transforms, Fourier transforms can be used to solve partial differentia equations. Again, it is inverting the transform to find the final solution where problems tend to lie. Consider again the diffusion equation and take Fourier transforms with respect to x to give

$$-Dk^2\overline{T} = \frac{\partial\overline{T}}{\partial t}$$

Let $T(x,0) = T_0(x)$ so integrating and using this,

$$\overline{T} = \overline{T}_0 e^{-Dk^2 t}$$

The most difficult part of the problem, as ever, is to invert the transform. Perhaps the easiest way to do this is use standard transforms and convolution. Recall that

$$\begin{aligned}
f * g &= \int_{-\infty}^{\infty} f(y)g(x-y)\,\mathrm{d}y\\
\mathcal{F}(f*g) &= \mathcal{F}(f)\,\mathcal{F}(g)
\end{aligned}$$

where the Fourier transform is being taken with respect to x. The solution to the differential equation is now in the form $\mathcal{F}(T_0)\,\mathcal{F}(g)$ where g is to be determined. Recalling that

$$\mathcal{F}^{-1}\left(e^{-Dk^2 t}\right) = \frac{\exp\left(\frac{-x^2}{4Dt}\right)}{\sqrt{4\pi Dt}}$$

it is evident that the solution is

$$T(x,t) = \int_{-\infty}^{\infty} T_0(y)\frac{\exp\left(\frac{-(x-y)^2}{4Dt}\right)}{\sqrt{4\pi Dt}}\,\mathrm{d}y$$

[†]Like the Pope but different.

(24.1.9) The Maximum Principle

Theorem 31 (The Maximum Principle For Laplace's Equation) *Let D be a connected bounded open set in 2 or 3 dimensions. Let $u(\mathbf{x})$ be a continuous harmonic function in D. Then the maximum and minimum values of u are attained on the boundary of D and nowhere inside D (unless u is constant).*

Proof. Working only in 2 dimensions, a local maximum is attained when $\frac{\partial^2 u}{\partial x^2} \leqslant 0$ and $\frac{\partial^2 u}{\partial y^2} \leqslant 0$. Now, u is harmonic and so obeys Laplace's equation, $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ everywhere in D. Let $v(\mathbf{x}) = u(\mathbf{x}) + \varepsilon|\mathbf{x}|^2$. Then

$$\begin{aligned}\nabla^2 v &= \nabla^2 u + \varepsilon \nabla^2 |\mathbf{x}|^2 \\ &= \varepsilon \nabla^2 |\mathbf{x}|^2 \\ &= 4\varepsilon \quad \text{(or } 6\varepsilon \text{ in 3 dimensions)}\end{aligned}$$

But at a local maximum $\nabla^2 v = 0$ and hence v can have no such point in D and so must attain its maximum at a point $\mathbf{x}_0$, say, on the boundary of D. Now,

$$u(\mathbf{x}) < v(\mathbf{x}) \leqslant v(\mathbf{x}_0) = u(\mathbf{x}_0) + \varepsilon|\mathbf{x}_0|^2$$

which in turn is less than the maximum value of u on the boundary of D plus εl^2 where l is the greatest distance from any point in D to the origin. Since this holds for all $\mathbf{x} \in D$ it must be the case that u attains its maximum on the boundary of D. A similar argument shows the minimum to be attained on the boundary also. □