Chapter 33

MSMYM1 Mathematical Linear Programming

(33.1) The Simplex Algorithm

The Simplex method for solving linear programming problems has already been covered in Chapter **??**. A given problem may always be expressed in the 'standard' form

$$f(\mathbf{x}) \to \min \qquad A\mathbf{x} = \mathbf{b} \qquad x_j \geqslant 0 \;\forall j \tag{1}$$

Say $\mathbf{x}$ is of length n and that there are m constraint equations ($n > m$), then the solution region is a convex subset of $\mathbb{R}^n$ (see Chapter **??** for a discussion of convex sets):

Theorem 2 *The set of feasible solutions to a linear programming problem in standard form with m equations in n unknowns is a convex subset of* $\mathbb{R}^n$.

Theorem 3 *The point* $\mathbf{x} \in \mathbb{R}^n$ *is a basic feasible solution of (1) if and only if it is an extreme point of the solution region.*

Of the n variables m need to be chosen as a basis, and thus there are $\binom{n}{m}$ basic feasible solutions.

Theorem 4 *The optimal solution occurs at an extreme point, and if it occurs at more than one extreme point* $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k$ *say then any convex linear combination of these extreme points is also optimal,* i.e.,

$$\sum_{i=1}^{k} \lambda_i \mathbf{x}_i \quad \textit{such that} \quad \sum_{i=1}^{k} \lambda_i = 1 \textit{ and } 0 \leqslant \lambda_i \leqslant 1 \textit{ for all } 1 \leqslant i \leqslant k$$

For example, if (x_1, y_1) and (x_2, y_2) are both optimal solutions to a linear programming problem with 2 variables, then any point on the line joining (x_1, y_1) and (x_2, y_2) is also optimal.

(33.1.1) Optimality

In the Simplex Algorithm optimality is checked using, not suprisingly, the "optimality cirterion". This is actually one of four criteria, depending on whether the problem is maximisation or minimisation, and how the Simplex Table has been constructed. To this end it is prudent to clarify the situation. (Note that $z_j = \mathbf{c}_B^\top B^{-1} \mathbf{A}_j$.)

- If the linear programming problem is maximisation:

- If the top row of the Simplex table is for $-f(\mathbf{x})$ then it tabulates values of $c_j - z_j$ and at optimality these are negative (or zero).
- If the top row of the Simplex table is for $f(\mathbf{x})$ then it tabulates values of $z_j - c_j$ and at optimality these are positive (or zero).

- If the linear programming problem is minimisation:
  - If the top row of the Simplex table is for $-f(\mathbf{x})$ then it tabulates values of $c_j - z_j$ and at optimality these are positive (or zero).
  - If the top row of the Simplex table is for $f(\mathbf{x})$ then it tabulates values of $z_j - c_j$ and at optimality these are negative (or zero).

Unfortunately there is no consistency as to which form is used, so these must be memorised. Note this mat make more sense if read after Section 33.2.2.

(33.1.2) Finding An Initial Basic Feasible Solution

As already mentioned, there are some $\binom{n}{m}$ basic feasible solutions. It is possible to find the optimal solution by exhaustively trying all these solutions, though clearly this is not a satisfactory method. (This is called the complete enumeration technique.)

The Simplex method is much preferable, though it is not clear how to find an initial basic feasible solution. It would be possible to use the complete enumeration technique until a basic feasible solution is found, but clearly this is not satisfactory. A sure-fire method is required.

To write a linear programming problem in standard form (with equality in the constraint equations) it is necessary to introduce slack variables. In the case $\mathbf{a}_i\mathbf{x} \leqslant b_i$ a slack variable s_i will be added to give

$$a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n + s_i = b_i$$

This constraint can be satisfied by putting $\mathbf{x} = \mathbf{0}$ and $s_i = b_i$. However, with constraints of the form $\mathbf{a}_i\mathbf{x} \geqslant b_i$ the slack variable is subtracted and as it must be positive this trick does not work. Similarly, when $\mathbf{a}_i\mathbf{x} \geqslant b_i$ no slack variable is added.

To overcome this problem for each constraint equation of the form $\mathbf{a}_i\mathbf{x} \geqslant b_i$ or $\mathbf{a}_i\mathbf{x} \geqslant b_i$ introduce an artificial variable r_i. Hence, for example,

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \dots a_{1n}x_n \leqslant b_1 \quad &\text{becomes} \quad a_{11}x_1 + a_{12}x_2 + \dots a_{1n}x_n + s_1 = b_1 \\
a_{21}x_1 + a_{22}x_2 + \dots a_{2n}x_n \geqslant b_2 \quad &\text{becomes} \quad a_{11}x_1 + a_{12}x_2 + \dots a_{1n}x_n - s_2 + r_1 = b_2 \\
a_{31}x_1 + a_{32}x_2 + \dots a_{3n}x_n = b_3 \quad &\text{becomes} \quad a_{31}x_1 + a_{32}x_2 + \dots a_{3n}x_n + r_2 = b_3
\end{aligned}$$

In this form the problem has a trivial feasible solution: where an artificial variable is added, $r_i = b_i$, and where no artificial variable is needed $s_i = b_i$. The task now is to remove the artificial variables from the solution, as they are meaningless. To do this one may procede with either the M-technique or the two phase method. In both cases the idea is to modify the objective function so that the artificial variables are highly undesirable.

M-technique

Having added slack and artificial variables use the new objective function $\mathbf{c}^\top\mathbf{x} + \sum Mr_i$. For maximisation $M < 0$ and for minimisation $M > 0$. This can now be used in the Simplex method, but in filling in the first table the cost coefficients of the basic variables are not all zero: some are M. In order to aviod this the cosntrtaint equations can be used to substitute into the objective function and so eliminate the artificial variables.

When running the Simplex algorithm many of the costs will be in terms of M. In choosing the pivot the column in which the relative cost is least (for minimisation, most for maximisation) is chosen to pivot with. The value of M is taken to be so large that there is a clear choice. Note that M is *not* a parameter: no case analysis arrises.

If the artificial variables cannot be eliminated *i.e.*, optimality is reached and the optimal solution contains an artificial variable then the original problem was infeasible.

Two Phase Method

Rather than modifying the existing objective function as in the M-technique, the two phase method uses an auxiliary objective function $\sum r_i$.

As in the M-technique the constraint equations must be used to elimitate the r_i from the objective function so that the Simplex algorithm can be used. Running the Simplex algorithm produces a basic feasible solution in the non-artificial variables: the optimal value is 0. This is the first phase.

The second phase uses the basic feasible solution found by the first phase with the original opbjective function. However, note that it will be necessary to use the constraint equations (without the artificial variables) to eliminate the basic variables from the objective function. With this done the Simples algorithm is run a second time to give the optimal solution.

(33.1.3) Cycling, Degeneracy, and Finiteness

To find which row to pivot on the ratios $\frac{\overline{b}_i}{a_{ij}}$ are compared for the chosen column j. If two rows have the same ratio then it is not clear which row should be chosen.

Suppose rows i and k have the same ratio. Then pivoting on row i gives

$$\overline{b}_k - a_{kj}\frac{\overline{b}_i}{a_{ij}} = \overline{b}_k - a_{kj}\frac{\overline{b}_k}{a_{kj}} = 0$$

and hence in the next iteration the solution becomes degenerate. Now, the degenerate variable may be chosen to leave the basis but could equally well be replaced by another variable taking value 0, thus producing another degenerate solution and having no effect on the value of the objective function. Continuing in this manner gives an opportunity for cycling.

In the degenerate case, cycling can occur because there are many feasible solutions with the same objective function value. Similarly, cycling may occur when there are multiple optimal solutions: obviously these need not be degenerate.

As already mentioned, when there are multiple optimal solutions—at extrema of the solution region—a family of optimal solutions can be constructed. Say that $\mathbf{x}$ is of length n, there are m constraint inequalities, and that k slack and artificial variables are introduced. Now,

- The solution is in $\mathbb{R}^n$
- Extreme points of the solution region occur at the intersection of n of the constraint inequalities. This includes the constraints $x_i \geqslant 0$.

It is possible that more than m constraints and non-negativity constraints meet at an extreme point. When this occurs, more than one basis gives the same solution: cycling occurs.

Cycling caused by degeneracy can be avoided. The method involves perturbing the b_is so that in subsequent Simplex tables the $\overline{b}_i$s will be different and so aviod a tie. Firstly, a rather technical lemma is needed.

Lemma 5 *If the first non-zero coefficient of a polynomial* $p(x) = a_0 + a_1 x + \cdots + a_n x^n$ *is greater than zero then* $\exists h > 0$ *such that* $0 < x < h$ *implies* $p(x) > 0$.

Proof. Let a_k be the first non-zero coefficient, and let

$$M = \max_{k \leqslant i \leqslant n} |a_i|$$

Note that a_k is the first non-zero coefficient of p, and assume that $x > 0$, then this gives

$$\begin{aligned} p(x) &= x^k \left(a_k + a_{k+1} x + \cdots + a_n x^{n-k} \right) \\ &\geqslant x^k \left(a_k - Mx - \cdots - Mx^{n-k} \right) \\ &\geqslant x^k \left(a_k - (m-k)Mx \right) \quad \text{for } 0 < x < 1 \end{aligned}$$

Now, this polynomial is positive if and only if

$$a_k > (m-k)Mx \qquad \Leftrightarrow \qquad x < \frac{a_k}{(m-k)M}$$

Hence choosing $h < \min\left(1, \frac{a_k}{(m-k)M}\right)$ the result is shown. □

Now, consider a linear programming problem, which by the addition of slack or artificial variables may be expressed in the form

$$\begin{aligned} x_1 + x_2 + \cdots + x_n &\to \min \\ a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n + x_{n+1} &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n + x_{n+2} &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n + x_{n+m} &= b_m \end{aligned}$$

For each constraint equation replace b_i by $b_i + \varepsilon^i$, then the initial solution $x_{n+i} = \varepsilon^i$ for $1 \leqslant i \leqslant m$ and $x_j = 0$ otherwise is clearly non-degenerate. Furthermore, putting $" = \mathbf{0}$ gives the original problem. The new problem may be written as

$$\mathbf{c}^\top \mathbf{x} \to \min \quad \text{such that} \quad A\mathbf{x} = \mathbf{b} + "$$

The initial basis will typically be $\mathbf{x}_B = (x_{m+1}, x_{m+2}, \ldots, x_n)$ so let B be the $n \times n$ submatrix of A with columns corresponding to these initial basic variables and let $B^{-1} = (\beta_{ij})$. Furthermore, let A_N be the matrix formed by the columns of A corresponding to non-basic variables, $\mathbf{x}_N$. Then

$$\begin{aligned} A\mathbf{x} &= \mathbf{b} \\ \begin{pmatrix} B & A_N \end{pmatrix} \begin{pmatrix} \mathbf{x}_B \\ \mathbf{x}_N \end{pmatrix} &= \mathbf{b} \\ B\mathbf{x}_B + A_N\mathbf{x}_N &= \mathbf{b} \quad \text{but } B \text{ is non-singular, so} \\ \mathbf{x}_B + B^{-1}A_N\mathbf{x}_N &= B^{-1}\mathbf{b} \end{aligned}$$

This system is that which occurs after pivoting: the basic variables have identity matrix columns in the cost coefficients matrix. The new value of $\mathbf{b}$

$$\overline{\mathbf{b}} = B^{-1}\mathbf{b} \quad \text{so} \quad \overline{b}_i = \sum_{j=1}^{m} \beta_{ij} b_j$$

Return now to considering the perturbed problem. This gives

$$\overline{b}_i(\varepsilon) = \sum_{j=1}^{m} \beta_{ij}(b_j + \varepsilon^j) = \overline{b}_i + \sum_{j=1}^{m} \beta_{ij}\varepsilon^j$$

In the simplex table the β_{ij}s are the entries of the matrix whose columns are those of the original feasible solution. In the first table this is always the identity matrix, though in subsequent iterations it becomes non trivial.

Returning now to the original problem, the pivoting column has been chosen (j say) and now the pivot θ is selected as

$$\theta = \min \left\{ \frac{b_i}{a_{ij}} \mid a_{ij} > 0 \text{ and } x_i \in \mathbf{x}_B \right\}$$

Say there is a tie on rows r and s. In the perturbed problem consider the comparrison

$$\frac{\beta_{r0}}{a_{rj}} \gtreqqless \frac{\beta_{s0}}{a_{sj}} \tag{6}$$

that is, the ratio except calculated with numerator as the number under the first initial basic variable in the corresponding row. Now,

$$\sum_{j=1}^{m} \beta_{sj}\varepsilon^j - \sum_{j=1}^{m} \beta_{rj}\varepsilon^j$$

is again a polynomial, and so by Lemma 5 if the first non-zero coefficient is greater than 0 then the polynimoal is positive for sufficiently small ε. Therefore make the comparrison in equation (6) for increasing values of j until inequality is reached. The lesser value is then chosen to for the row of the pivot.

(33.1.4) Revised Simplex Method

Overview

The Simplex method involves many calcuations, and while the final Simplex table is useful the intermediate ones are of negligible interest. The revised Simplex method reduces the ammount of calculations required by using only the 'business end' of the simplex table and disregarding the rest. Given a linear programming problem in standard form,

$$\begin{aligned} z &= \mathbf{c}^\top \mathbf{x} \to \max \\ \begin{pmatrix} A & I \end{pmatrix} \mathbf{x} &= \mathbf{b} \\ \mathbf{x} &\geqslant \mathbf{0} \end{aligned}$$

The variables of the problem are partitioned as follows.

- Let $\mathbf{x}^{(1)}$ be the original variables of the problem *i.e.*, no slack or artificial variables.

- Let $\mathbf{x}^{(2)}$ be the initial basic variables (which will be selected from the slack and artificial variables)

No variables will be missing from this decomposition of $\mathbf{x}$ since all the artificial variables are used in the initial basis. Let $\mathbf{c}^{(1)}$ and $\mathbf{c}^{(2)}$ be the corresponding cost vectors. The problem may then be written in the form

$$\begin{pmatrix} 1 & -(\mathbf{c}^{(1)})^\top & -(\mathbf{c}^{(2)})^\top \\ 0 & A & I \end{pmatrix} \begin{pmatrix} z \\ \mathbf{x}^{(1)} \\ \mathbf{x}^{(2)} \end{pmatrix} = \begin{pmatrix} 0 \\ \mathbf{b} \end{pmatrix}$$

Let $\begin{pmatrix} A & I \end{pmatrix} = A'$ and have jth column $\mathbf{A}'_j$. In the usual way A' may be partitioned as A'_B and A'_N corresponding to basic and non-basic variables so that

$$A'\mathbf{x} = \mathbf{b} \quad \Leftrightarrow \quad \begin{pmatrix} A'_B & A'_N \end{pmatrix} \begin{pmatrix} \mathbf{x}_B \\ \mathbf{x}_N \end{pmatrix} = \mathbf{b}$$

From this

$$\begin{aligned} \mathbf{x}_B &= (A'_B)^{-1}\mathbf{b} - (A'_B)^{-1}A'_n\mathbf{x}_n \\ z &= \mathbf{c}_B^\top \mathbf{x}_B + \mathbf{c}_N^\top \mathbf{x}_N \\ &= \mathbf{c}_B^\top \left((A'_B)^{-1}\mathbf{b} - (A'_B)^{-1}A'_n\mathbf{x}_n \right) + \mathbf{c}_N^\top \mathbf{x}_N \\ &= \mathbf{c}_B^\top (A'_B)^{-1}\mathbf{b} - \left(\mathbf{c}_B^\top (A'_B)^{-1}A'_n - \mathbf{c}_N^\top \right) \mathbf{c}_N \end{aligned}$$

But for any given basis $\mathbf{x}_N = \mathbf{0}$ and thus

$$\begin{aligned} \mathbf{x}_B &= (A'_B)^{-1}\mathbf{b} \\ z &= \mathbf{c}_B^\top (A'_B)^{-1}\mathbf{b} \\ &= \mathbf{c}_B^\top \mathbf{x}_B \end{aligned}$$

	z	$\mathbf{x}^{(1)}$	$\mathbf{x}^{(2)}$	Solution
z	1	$\mathbf{c}_B^\top B^{-1}A - \mathbf{c}^{(1)}$	$\mathbf{c}_B^\top B^{-1} - \mathbf{c}^{(2)}$	$\mathbf{c}_B^\top B^{-1}\mathbf{b}$
$\mathbf{x}_B$	$\mathbf{0}$	$B^{-1}A$	B^{-1}	$B^{-1}\mathbf{b}$

Table 1: The Simplex table at any stage.

Putting $A'_B = B$, this can be expressed in matrix form as

$$\begin{aligned}
\begin{pmatrix} 1 & -\mathbf{c}_B^\top \\ 0 & B \end{pmatrix} \begin{pmatrix} z \\ \mathbf{x}_B \end{pmatrix} &= \begin{pmatrix} 0 \\ \mathbf{b} \end{pmatrix} \\
\Rightarrow \begin{pmatrix} z \\ \mathbf{x}_B \end{pmatrix} &= \begin{pmatrix} 1 & -\mathbf{c}_B^\top \\ 0 & B \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ \mathbf{b} \end{pmatrix} \\
&= \begin{pmatrix} 1 & -\mathbf{c}_B^\top B^{-1} \\ 0 & B^{-1} \end{pmatrix} \begin{pmatrix} 1 & -(\mathbf{c}^{(1)})^\top & -(\mathbf{c}^{(2)})^\top \\ 0 & A & I \end{pmatrix} \begin{pmatrix} z \\ \mathbf{x}^{(1)} \\ \mathbf{x}^{(2)} \end{pmatrix} \\
&= \begin{pmatrix} 1 & \mathbf{c}_B^\top B^{-1}A - \mathbf{c}^{(1)} & \mathbf{c}_B^\top B^{-1} - \mathbf{c}^{(2)} \\ 0 & B^{-1}A & B^{-1} \end{pmatrix} \begin{pmatrix} z \\ \mathbf{x}^{(1)} \\ \mathbf{x}^{(2)} \end{pmatrix}
\end{aligned}$$

Hence for any given basis $\mathbf{c}_B$ the state of the system *i.e.,* the components of the Simplex table can be expressed in terms of the initial basis and original variables. The Simplex table will be of the form given in Table 1.

From these calcuations it is apparent that any Simplex table can be found, given the initial state of the system, and the matrix B^{-1}.

Updating B^{-1}

In the case of the Simplex method at some general iteration a basic variable x_r is removed in favour of some non-basic variable x_s say. It can be shown that B^{-1} is updated as

$$B^{-1} := EB^{-1} \quad \text{where} \quad E = \begin{pmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \dots & \mathbf{e}_{r-1} & \mathbf{K} & \mathbf{e}_{r+1} & \dots & \mathbf{e}_m \end{pmatrix}$$

where $\mathbf{e}_i$ is the ith standard ordered basis vector and $\mathbf{K}$ is calcualted as follows. Calculate $\overline{\mathbf{A}}_s = B^{-1}\mathbf{A}_s$ and let

$$\mathbf{K} = \begin{pmatrix} \frac{-\bar{a}_{1s}}{\bar{a}_{rs}} & \frac{-\bar{a}_{2s}}{\bar{a}_{rs}} & \dots & \frac{-\bar{a}_{(r-1)s}}{\bar{a}_{rs}} & \frac{1}{\bar{a}_{rs}} & \frac{-\bar{a}_{(r+1)s}}{\bar{a}_{rs}} & \dots & \frac{-\bar{a}_{ms}}{\bar{a}_{rs}} \end{pmatrix}^\top$$

The remaining question is as to which variable should leave the basis and which should enter *i.e.,* the values of r and s. To determine the variable to enter the basis the optimality criterion is used.

Variable Entering The Basis

As usual the variable to enter the basis is that which violates the optimality criterion. The reduced cost $\mathbf{c}_B^\top B^{-1}\mathbf{A}_j - c_j$ is examined, and a matrix of these can be computed as

$$(\mathbf{z} - \mathbf{c})_N = \begin{pmatrix} -1 & \mathbf{c}_B^\top B^{-1} \end{pmatrix} \begin{pmatrix} -\mathbf{c}_N^\top \\ A_N \end{pmatrix}$$

where $\mathbf{c}_N^\top$ is the row vector of cost coefficients of the non-basic variables, and A_N is the matrix of columns of A of the non-basic variables.

Taking the minimum (for a maximisation problem) or maximum (for a minimisation problem) will reveal the index of the non-basic variable that will enter the basis.

Variable Leaving The Basis

The variable leaving the basis is chosen so that no basic variable becomes negative and the leaving variable becomes zero. Let x_s be the variable that enters the basis. The pivot is computed as

$$\theta = \min_{i \text{ basic}} \left\{ \frac{(B^{-1}\mathbf{b})_i}{(B^{-1}\mathbf{A}_s)_i} \mid (B^{-1}\mathbf{A}_s)_i > 0 \right\}$$

(33.2) Duality

(33.2.1) Constructing The Dual

Noting that the dual to the dual is the primal, writing a linear programming problem in a suitable form allows its dual to be formed from Table 2.

Primal Problem	Description Of Indices	Dual Problem
$\mathbf{c}^T\mathbf{x} \to \min$		$\mathbf{b}^T\text{ß} \to \max$
$\mathbf{a}_i^T\mathbf{x} = b_i$	$i \in I$	π_i free
$\mathbf{a}_i^T\mathbf{x} \geqslant b_i$	$i \in \overline{I}$	$\pi_i \geqslant 0$
$x_j \geqslant 0$	$j \in J$	$\text{ß}^T\mathbf{A}_j \leqslant c_j$
x_j free	$j \in \overline{J}$	$\text{ß}^T\mathbf{A}_j = c_j$

Table 2: Correspondence between the primal and dual linear programming problems.

It is necessary to write a linear programming problem in one of these forms before attempting to determine the dual (note the dual of the fual is the primal, so Table 2 can be used 'backwards').

(33.2.2) Connections Between The Dual And The Primal

A linear programming problem is closely connected to its dual, and a solution to one gives information about the solution of the other. This ins encapsulated in the duality theorems, and is exploited in the Primal-Dual algorithm (see Section 33.2.2).

Theorem 7 (Weak Duality Theorem) $\forall \mathbf{x} \in M_P \quad \forall \text{ß} \in M_D \quad \mathbf{c}^T\mathbf{x} \geqslant \text{ß}^T\mathbf{B}$ i.e., *for all feasible solutions,* $f(\mathbf{x}) \geqslant \phi(\text{ß})$.

Proof. Now, specifically for $j \in J$, $c_j \geqslant \text{ß}^T\mathbf{A}_j$. However, when $j \in \overline{J}$ there is equality, so the relationship holds in general. Similarly $b_i \leqslant \mathbf{a}_i^T\mathbf{x}$. Hence,

$$\begin{aligned} f(\mathbf{x}) = \mathbf{c}^T\mathbf{x} &= \sum_j c_j x_j & \phi(\text{ß}) = \text{ß}^T\mathbf{b} &= \sum_i \pi_i b_i \\ &\geqslant \sum_j \text{ß}^T\mathbf{A}_j x_j & &\leqslant \sum_i \pi_i \mathbf{a}_i^T\mathbf{x} \\ &= \text{ß}^T A\mathbf{x} & &= \text{ß}^T A\mathbf{x} \end{aligned}$$

□

Theorem 8 *If $\mathbf{x}$ is an optimal solution to a linear programming problem and $f(\mathbf{x}) = \mathbf{c}^\top \mathbf{x}$ is finite, then an optimal solution to the dual is $\mathbf{c}_B^\top B^{-1}$.*

Proof. Note the ambiguity of notation where B^{-1} is a matrix and B is the set of basic indices. Now, for $j \notin B$, $x_j = 0$ and hence for any solution $\mathbf{x}$ of the primal,

$$\begin{aligned} f(\mathbf{x}) &= \mathbf{c}_B^\top B^{-1}\mathbf{b} - \sum_{j \notin B} (\mathbf{c}_B^\top B^{-1}\mathbf{A}_j - c_j)x_j \\ &= \mathbf{c}_B^\top B^{-1}\mathbf{b} - \sum_{j \notin B} (z_j - c_j)x_j \end{aligned}$$

where z_j is known as the "simplex multiplier" and $c_j - z_j$ is known as the "reduced cost" or "relative cost". If B is the set of basic indices for an optimal solution then

$$z_j - c_j = \begin{cases} 0 & \text{if } j \in B \\ \mathbf{c}_B^\top b^{-1}\mathbf{A}_j - c_j & \text{if } j \notin B \end{cases}$$

Hence

$$\begin{aligned} z_j - c_j \geqslant 0 \;\Leftrightarrow\; & \mathbf{c}_B^\top B^{-1}\mathbf{A}_j - c_j \geqslant 0 \\ \Leftrightarrow\; & \begin{cases} \mathbf{c}_B^\top B^{-1}\mathbf{A}_j - c_j \geqslant 0 & \text{if } j \notin B \text{ and is the index of an original variable} \\ \mathbf{c}_B^\top B^{-1} & \text{if } j \notin B \text{ and is the index of a slack or artificial variable} \end{cases} \end{aligned}$$

because if j is the index of a slack or artificial variable that was in the original basis then $\mathbf{A}_j = \mathbf{e}_j$ and $c_j = 0$. Hence

$$\mathbf{c}_B^\top B^{-1}\mathbf{A}_j \geqslant c_j \quad \text{and} \quad \mathbf{c}_B^\top b^{-1} \geqslant 0$$

i.e., $\mathbf{c}_B^\top B^{-1}$ is a feasible solution to the dual. But

$$f(\mathbf{x}) = \mathbf{c}^\top \mathbf{x} = \mathbf{c}_B^\top \mathbf{x}_B = \mathbf{c}_B^\top B^{-1}\mathbf{b}$$

which is the value of the objective function of the dual. Since this coincides with the value of the objective function of the primal Theorem 7 (Weak Duality) shows that $\mathbf{c}_B^\top B^{-1}$ is an optimal solution to the dual. □

The "reduced costs" are located in the final simplex table, as shown in Table 1. For the basic variables they are all zero and for the non-basic variables they are positive. By the proof of Theorem 8 the optimal Simpex table for the primal can be used to calculate the optimal solution to the dual. Take the "reduced costs" in the original basic positions, then

$$\pi_j = \mathbf{c}_B^\top B^{-1}\mathbf{e}_j = z_j(z_j - c_j) + c_j$$

Usually the costs c_j are zero, so that the solution can be read off from the tabe easily.

The following important result now follows readily.

Theorem 9 (Strong Duality) *If a linear programming problem or its dual has a finite solution then so does the other and the optimal values are equal.*

Proof. Immediate from Theorem 8. □

(33.2.3) Interpreting Dual Variables

As well as the strong mathematia relationship between a linear programming problem and its dual, there is also a connection in the interpretation of the variables and constraints.

Does anybody know what it is?

Section 33.3 gives methods for analysing how changes in the original problem effect the solution. However, the relationship between a linear pprogramming problem and its dual allows the identification of the constraints which will effect the solution: clearly, in practical situations the constraints effecting the solution will be of particular itnerest.

Consider a linear programmig problem and its dual

$$\begin{aligned} \mathbf{c}^\top \mathbf{x} &\to \max \\ A\mathbf{x} &\leqslant \mathbf{b} \\ \mathbf{x} \geqslant \mathbf{0}&\mathbf{y} \end{aligned} \qquad\qquad \begin{aligned} y_0 = \mathbf{y}^\top \mathbf{b} &\to \min \\ A^\top \mathbf{y} &\geqslant \mathbf{c} \\ &\geqslant \mathbf{0} \end{aligned}$$

Consider changing b_i to $b'_i = b_i + \Delta b_i$. Then

$$y'_0 = y_0 + \sum_{i=1}^{m} y_i \Delta b_i$$

Also, under this transformation the primal solution will remain optimal since $z_j - c_j = \mathbf{c}_B^\top B^{-1}\mathbf{A}_j - c_j$ is not effected by changes in $\mathbf{b}$. However, the feasibility of the solution may change since it is not necessarily the case that $\mathbf{x}_B = B^{-1}\mathbf{b}' \geqslant \mathbf{0}$.

Certainly the solution will still be feasible for a sufficiently small change, *i.e.*, $\exists' \varepsilon > 0$ such that if $\|\Delta \mathbf{b}\| < \varepsilon$ then $B^{-1}(\mathbf{b} + \Delta \mathbf{b}) \geqslant 0$.

Now, y_i is the change in objective function per unit change in b_i, so that roughly speaking

$$\frac{\partial y_0}{\partial b_i} = y_i$$

This may be used in a number of ways.

- If $y_i > 0$ one could seek to increase b_i in order to increase the value of the objective function.
- If i is the index for which y_i is the largest positive dual variable then increasing b_i will give the most increase in the objective function.

While doing this it is required to maintain feasibility.

Complementary Slackness Theorem

(33.2.4) Dual Simplex Method

The simplex method works by finding a sequence of feasible solutions with non-decreasing deviation from optimality. Alternatively, roughly speaking, a sequence of optimal solutions with non-decreasing feasibility could be found. This is what the dual simplex method does. Consider a linear programming problem and its dual,

$$\begin{aligned} \mathbf{c}^\top \mathbf{x} &\to \min \\ A\mathbf{x} &\leqslant \mathbf{b} \\ \mathbf{x} &\geqslant \mathbf{0} \end{aligned} \qquad\qquad \begin{aligned} \mathbf{y}^\top \mathbf{b} &\to \max \\ A^\top \mathbf{y} &\geqslant \mathbf{c} \\ \mathbf{y} &\geqslant \mathbf{0} \end{aligned}$$

Now consider changing $\mathbf{b}$ to another vector $\mathbf{b}^{(1)}$ say. Modifying the final Simplex table by entering $\mathbf{x}_B = B^{-1}\mathbf{b}^{(1)}$ instead of $\mathbf{x}_B = B^{-1}\mathbf{b}$, if $B^{-1}\mathbf{b}^{(1)} \geqslant \mathbf{0}$ then the solution is still feasible. As the reduced costs have not been changed the table now gives an optimal solution to the modifieed problem.

If however $B^{-1}\mathbf{b}^{(1)}$ has negative entries then the solution is nolonger feasible. However, $\mathbf{c}_B^\top$ is still a feasible solution to the dual though it will not be optimal. The Simplex algorithm may therefore be run on the dual and in fact this can be done in the table for the primal: this constitutes the dual Simplex algorithm. Let $(B^{-1}\mathbf{b}^{(1)})_i = \bar{b}_i$ and $\bar{c}_j = c_j - \mathbf{c}_B^\top B^{-1}\mathbf{A}_j$.

1. Select for the pivoting row r where $\bar{b}_r = \min\{\bar{b}_i \mid \bar{b}_i < 0\}$. Note that since $\mathbf{x}_B = B^{-1}\mathbf{b}^{(1)}$ row r is that of the most negative basic variable.

2. For any column k, pivoting arround $\bar{a}_{rk}$ (where the bar denotes that the value is in the previously optimal Simplex table, not the original problem) optimality must be maintained. Let $\bar{c}_j'$ be the value replacing $\bar{c}_j$ after pivoting. To maintain optimality it is required that

$$\begin{aligned} \bar{c}_j' &\geqslant 0 \\ \bar{c}_j - \bar{c}_k \frac{\bar{a}_{rj}}{\bar{a}_{rk}} &\geqslant 0 \end{aligned}$$

and this must be so for all j.

(33.3) Post Optimality Analysis

Having obtained an optimal solution it is often of interest as to how the optimal solution would change depending on changes in $\mathbf{b}$, $\mathbf{c}$, or in A. If only few changes are made then it is possible to re-calculate an optimal solution without much work.

It is possible to consider changes in single values, in which case a single iteration of the Simplex method (or Dual Simples method) is often sufficient to find a new optimal feasible solution. The approach adopted here is to consider many changes at once, seeking bounds within which the solution remains feasible and optimal.

(33.3.1) Changes In c

Say $\mathbf{c} \mapsto \mathbf{c} + \theta\mathbf{E}$ for some $\mathbf{E} \in \mathbb{R}^n$. Now, the solution is calculated as $B^{-1}\mathbf{b}$ which is not effected by changes in $\mathbf{c}$, hence feadibility is not effected by changes in $\mathbf{c}$.

Suppose that an optimal solution is known for $\theta = \alpha$ (frequently $\alpha = 0$) and consider $\theta = \beta$. The theory will be developed for $\beta > \alpha$ with comments on how to adapt for $\beta < \alpha$. Now,

$$\begin{aligned}\mathbf{c}^{(\theta)} &= \mathbf{c} + \theta\mathbf{E} \\ &= \mathbf{c} + (\theta - \alpha)\mathbf{E} + \alpha\mathbf{E} \\ &= \mathbf{c}^{(\alpha)} + (\theta - \alpha)\mathbf{E}\end{aligned}$$

Now put $z_j^{(\theta)} = (\mathbf{c}_B^{(\theta)})^\top (B^{(\alpha)})^{-1}\mathbf{A}_j$ and assume that the linear programming problem under consideration is maximisation with $z_j - c_j$ tabulated. Now,

$$\begin{aligned}z_j^{(\theta)} - c_j^{(\theta)} &= (\mathbf{c}_B^{(\theta)})^\top (B^{(\alpha)})^\top \mathbf{A}_j - \mathbf{c}_j^{(\theta)} \\ &= (\mathbf{c}_B^{(\alpha)})^\top (B^{(\alpha)})^\top \mathbf{A}_j + (\theta - \alpha)\mathbf{E}_B^\top (B^{(\alpha)})^\top \mathbf{A}_j - \mathbf{c}_j^{(\alpha)} - (\theta - \alpha)E_j \\ &= \left(z_j^{(\alpha)} - c_j^{(\alpha)}\right) + (\theta - \alpha)\left(\mathbf{E}_B^\top (B^{(\alpha)})^\top \mathbf{A}_j - E_j\right) \qquad (10)\end{aligned}$$

But $\theta - \alpha > 0$ and optimality is certainly maintained when

$$\mathbf{E}_B^\top (B^{(\alpha)})^\top \mathbf{A}_j - E_j \geqslant 0$$

(less than or equal to for $\theta < \alpha$ because $\theta - \alpha < 0$). Of course there may be certain values of j for which this criterion is not met then equation (10) gives

$$\begin{aligned}&0 \leqslant z_j^{(\theta)} - c_j^{(\theta)} \\ &0 \leqslant \left(z_j^{(\alpha)} - c_j^{(\alpha)}\right) + (\theta - \alpha)\left(\mathbf{E}_B^\top (B^{(\alpha)})^\top \mathbf{A}_j - E_j\right) \\ &\theta \leqslant \alpha + \frac{-(z_j^{(\alpha)} - c_j^{(\alpha)})}{\mathbf{E}_B^\top (B^{(\alpha)})^{-1}\mathbf{A}_j - E_j} \quad \text{where } \mathbf{E}_B^\top (B^{(\alpha)})^{-1}\mathbf{A}_j - E_j < 0 \\ &\beta = \alpha + \min_{1\leqslant j\leqslant n}\left\{\frac{-(z_j^{(\alpha)} - c_j^{(\alpha)})}{\mathbf{E}_B^\top (B^{(\alpha)})^{-1}\mathbf{A}_j - E_j} \mid \mathbf{E}_B^\top (B^{(\alpha)})^{-1}\mathbf{A}_j - E_j < 0\right\}\end{aligned}$$

Note that for a minimisation problem with $z_j - c_j$ tabulated that $\theta - \alpha \leqslant 0$ and then

$$\beta = \alpha + \max_{1\leqslant j\leqslant n}\left\{\frac{-(z_j^{(\alpha)} - c_j^{(\alpha)}}{\mathbf{E}_B^\top (B^{(\alpha)})^{-1}\mathbf{A}_j - E_j} \mid \mathbf{E}_B^\top (B^{(\alpha)})^{-1}\mathbf{A}_j - E_j > 0\right\}$$

which is in fact the same as in the maximisation problem when $\theta < \alpha$. For $\theta > \beta$ the solution is nolonger optimal, and the Simplex method may be used to find a new optimal solution. This process may then be repeated to find at what point the new solution is nolonger optimal.

(33.3.2) Changes in b

Say $\mathbf{b} \mapsto \mathbf{b} + \theta\boldsymbol{ø}$ for some $\boldsymbol{ø} \in \mathbb{R}^m$. Now, the reduced costs are calculated as $\mathbf{c}_B^\top B^{-1}\mathbf{A}_j - c_j$ which is not effected by changes in **b**, hence optimality is not effected by changes in **b**.

Suppose that an optimal solution is known for $\theta = \alpha$ (frequently $\alpha = 0$) and consider $\theta = \beta$. The theory will be developed for $\beta > \alpha$ with notes on how to adapt for $\beta < \alpha$. Now,

$$\begin{aligned}\mathbf{b}^{(\theta)} &= \mathbf{b} + \theta\boldsymbol{\o} \\ &= \mathbf{b} + (\theta - \alpha)\boldsymbol{\o} + \alpha\boldsymbol{\o} \\ &= \mathbf{b}^{(\alpha)} + (\theta - \alpha)\boldsymbol{\o}\end{aligned}$$

Now put $\mathbf{x}_B^{(\theta)} = (B^{(\alpha)})^{-1}\mathbf{b}^{(\theta)}$ then

$$\begin{aligned}\mathbf{x}_B^{(\theta)} &= (B^{(\alpha)})^{-1}\mathbf{b}^{(\theta)} \\ &= (B^{(\alpha)})^{-1}(\mathbf{b} + \theta\boldsymbol{\o}) \\ &= (B^{(\alpha)})^{-1}\mathbf{b}^{(\alpha)} + (\theta - \alpha)(B^{(\alpha)})^{-1}\boldsymbol{\o} \qquad (11)\end{aligned}$$

But $\theta - \alpha > 0$ and therefore to maintian feasibility it is required that

$$\left((B^{(\alpha)})^{-1}\boldsymbol{\o}\right)_i \geqslant 0 \text{ for all basic indices } i$$

(this becomes less than or equal to when $\theta < \alpha$ because $\theta - \alpha < 0$). Of course there may be certain values of i for which this criterion is not met then equation (11) gives

$$\begin{aligned}\mathbf{x}_B^{(\theta)} &\geqslant \mathbf{0} \\ \left((B^{(\alpha)})^{-1}\boldsymbol{\o}\right)_i &\geqslant 0 \quad i \text{ basic} \\ \theta &\leqslant \alpha + \frac{-\left((B^{(\alpha)})^{-1}\mathbf{b}^{(\alpha)}\right)_i}{\left((B^{(\alpha)})^{-1}\boldsymbol{\o}\right)_i} \quad i \text{ basic and } \left((B^{(\alpha)})^{-1}\boldsymbol{\o}\right)_i < 0 \\ \beta &= \alpha + \min_{i \text{ basic}} \left\{ \frac{-\left((B^{(\alpha)})^{-1}\mathbf{b}^{(\alpha)}\right)_i}{\left((B^{(\alpha)})^{-1}\boldsymbol{\o}\right)_i} \mid \left((B^{(\alpha)})^{-1}\boldsymbol{\o}\right)_i < 0 \right\}\end{aligned}$$

There is no change in this condition for a minimisation problem, indeed no assumption was made about the kind of problem this was. However, for $\theta < \alpha$,

$$\beta = \alpha + \max_{i \text{ basic}} \left\{ \frac{-\left((B^{(\alpha)})^{-1}\mathbf{b}^{(\alpha)}\right)_i}{\left((B^{(\alpha)})^{-1}\boldsymbol{\o}\right)_i} \mid \left((B^{(\alpha)})^{-1}\boldsymbol{\o}\right)_i > 0 \right\}$$

When $\theta > \beta$ a new solution will have to be calculated, and since feasibility is violated this can be done with the dual Simplex method.

(33.3.3) Changes In A

Say $\mathbf{A}_j \mapsto \mathbf{A}_j + \theta\boldsymbol{\o}_j$ for some $\boldsymbol{\o}_j \in \mathbb{R}^m$. Now, the initial basis consists of variables that are not part of the original problem. Therefore B^{-1} is uneffected by changes in A. Since the solution is calculated as $B^{-1}\mathbf{b}$ (which is not effected by changes in $\mathbf{A}_j$) feasdibility is not effected by changes in $\mathbf{A}_j$.

Suppose that an optimal solution is known for $\theta = \alpha$ (frequently $\alpha = 0$) and consider $\theta = \beta$. The theory will be developed for $\beta > \alpha$ and notes made for how to adapt for $\theta < \alpha$. Now,

$$\begin{aligned}\mathbf{A}_j^{(\theta)} &= \mathbf{A}_j + \theta\boldsymbol{\o}_j \\ &= \mathbf{A}_j^{(\alpha)} + (\theta - \alpha)\boldsymbol{\o}_j\end{aligned}$$

Now put $z_j^{(\theta)} = \mathbf{c}_B^\top (B^{(\alpha)})^{-1}\mathbf{A}_j^{(\theta)}$ and assume that the linear programming problem under consideration is maximisation with $z_j - c_j$ tabulated. Now,

$$\begin{aligned} z_j^{(\theta)} - c_j &= \mathbf{c}_B^\top B^{-1}\mathbf{A}_j^{(\theta)} - c_j \\ &= \mathbf{c}_B^\top B^{-1}\left(\mathbf{A}_j^{(\alpha)} + (\theta-\alpha)\boldsymbol{\o}_j\right) \\ &= \left(\mathbf{c}_B^\top B^{-1}\mathbf{A}_j^{(\alpha)} - c_j\right) + \mathbf{c}_B^\top B^{-1}(\theta-\alpha)\boldsymbol{\o}_j \\ &= \left(z_j^{(\alpha)} - c_j\right) + (\theta-\alpha)\mathbf{c}_B^\top B^{-1}\boldsymbol{\o}_j \end{aligned} \tag{12}$$

But $\theta - \alpha > 0$ and therefore optimality is certainly maintained when

$$\mathbf{c}_B^\top B^{-1}\boldsymbol{\o}_j \geqslant 0$$

(this is less than or equal to for $\theta < \alpha$ because in this case $\theta - \alpha < 0$). Of course there may be certain values of j for which this criterion is not met then equation (12) gives

$$\begin{aligned} z_j^{(\theta)} - c_j &\geqslant 0 \\ \left(z_j^{(\alpha)} - c_j\right) + (\theta-\alpha)\mathbf{c}_B^\top B^{-1}\boldsymbol{\o}_j &\geqslant 0 \\ \theta &\leqslant \alpha + \frac{-(z_j^{(\alpha)} - c_j)}{\mathbf{c}_B^\top B^{-1}\boldsymbol{\o}_j} \quad \text{where } \mathbf{c}_B^\top B^{-1}\boldsymbol{\o}_j < 0 \\ \beta &= \alpha + \min_{1\leqslant j\leqslant n}\left\{\frac{-(z_j^{(\alpha)} - c_j)}{\mathbf{c}_B^\top B^{-1}\boldsymbol{\o}_j} \mid \mathbf{c}_B^\top B^{-1}\boldsymbol{\o}_j < 0\right\} \end{aligned}$$

Note that for a minimisation problem with $z_j - c_j$ tabulated that $\theta - \alpha < 0$ and then

$$\beta = \alpha + \max_{1\leqslant j\leqslant n}\left\{\frac{-(z_j^{(\alpha)} - c_j)}{\mathbf{c}_B^\top B^{-1}\boldsymbol{\o}_j} \mid \mathbf{c}_B^\top B^{-1}\boldsymbol{\o}_j > 0\right\}$$

For $\theta > \beta$ the solution is nolonger optimal, and the Simplex method may be used to find a new optimal solution. This process may be repeated to find at what point the new solution is nolonger optimal.

(33.3.4) Changes In **c**, In **b**, And In A

When **c**, **b**, and A are all changing, values of θ will be obtained for each. This gives rise to a case analysis.

(33.3.5) New Variables And New Constraint Equations

If a new variable is added, then it will only effect the solution if its inclusion in the basis could improve the value of the objective function *i.e.*, it violates the optimality criterion. For a new variable x_{n+1} calculate

$$z_j - c_j = \mathbf{c}_B^\top B^{-1}\mathbf{A}_{n+1}$$

where $\mathbf{A}_{n+1}$ are the coefficients of the new variable in the constraint equations. If this violates the optimality criterion then the Simplex method can be applied.

If an additional constraint is not violated then it has no effect on the solution. If however it is violated then a new solution must be calculated.

1. Write the new constraint in terms of non-basic variables. This can be done using the other constraints.
2. Add a slack variable to the new constraint, and add it to the previously optimal Simplex table in a new row, the slack variable taking the value of the right hand side of the new constraint inequality.
3. Using the Simplex method, the slack variable of the new constraint inequality will be removed from the basis in favour of some other variable. An optimal solution will be found.