Chapter 30

MSMYP1 Further Complex Variable Theory

(30.1) Multifunctions

A multifunction is a function that may take many values at the same point. Clearly such functions are problematic for an analytic study, though for the most part there is a convenient way to prevent their bad behaviour. Unfortunately some functions of $\mathbb{R} \to \mathbb{R}$ become multifunctions when extended to $\mathbb{C}$, so first of all these are dealt with.

(30.1.1) Arguments

Representing a complex number in the form $z = re^{i\theta}$, the argument of z is the number θ which gives the angle (in radians) that a line segment connecting the origin to z makes with the positive real axis. For a given complex number $z = re^{i\theta}$ there are many arguments, namely $\theta \pm 2k\pi$ for $k \in \mathbb{Z}$. Let

$$\mathrm{ARG}(z) = \left\{ \theta \mid z = re^{i\theta} \right\}$$

Any one of the elements of ARG(z) may be chosen for use, so let $\arg(z) \in \mathrm{ARG}(z)$, which may also be treat as a function

$$\arg(z) \colon \mathbb{C} \setminus \{0\} \to \mathbb{R}$$

$\arg(z)$ is called a choice for ARG(z), and no choice is a continuous function. This is simply demonstrated by noting that any continuous function is continuous when restricted to any contour. Choosing the unit circle as a contour,

$$\gamma \colon [0, 1) \to \mathbb{C} \quad \text{defined by} \quad \gamma \colon t \mapsto e^{it}$$

Plotting the function $\arg(z)$ on a third axis (see Figure 30.1.1) there is clearly a discontinuity where the function returns to the positive real axis.

The problems with the argument function occur when the positive real axis is crossed, and so can be solved by removing the positive real axis from $\mathbb{C}$. In fact any half-line can be removed, so define

$$L_\alpha = \left\{ z \in \mathbb{C} \mid z = re^{i\alpha} \ \forall r \geqslant 0 \right\}$$

In the cut plane $\mathbb{C} \setminus L_\alpha$ the choice

$$\arg(z) \in (\alpha, \alpha + 2\pi)$$

is continuous.

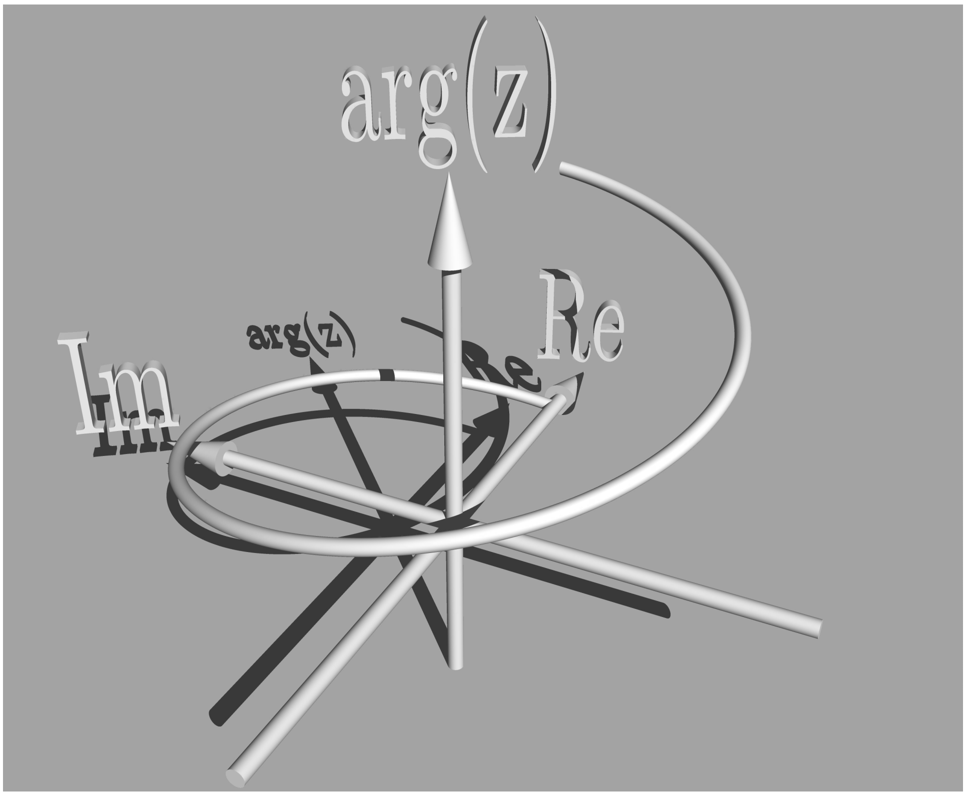

Figure 1: The argument function is not continuous.

(30.1.2) Logarithms

As complex numbers have expressions in terms of exponentials it is important that any definition of a logarithm is consistent with this.

$$\begin{aligned} z &= |z|e^{i\arg(z)} \\ &= e^{\ln|z|}e^{i\arg(z)} \\ &= \exp\left(\ln|z| + i\arg(z)\right) \end{aligned}$$

Hence make the definition

$$lnz = ln|z| + i\arg(z)$$

The appearance of the argument function makes this a multifunction. Similarly to the argument, let

$$\mathrm{LOG}(z) = \{\ln|z| + i\theta \mid \theta \in \mathrm{ARG}(z)\}$$

However, in the cut plane $\mathbb{C} \setminus L_\alpha$, $\ln z$ is a sum of continuous functions and thus is continuous. In fact $\ln z$ is analytic. Each possible cut of the complex plane gives rise to an analytic logarithm function, all of which are different. It is therefore common to speak of an "analytic branch of the logarithm" corresponding to a particular cut. Of course, it is most convenient to use the cut $\mathbb{C} \setminus L_0$.

(30.1.3) Powers

For $w, z \in \mathbb{C}$

$$z^w = \exp(w \ln z)$$

and thus exponentiation is a multifunction. Obviously cutting the plant will eliminate this problem.

(30.2) Poles And Zeros

The zeros and poles of a function $f\colon \mathbb{C} \to \mathbb{C}$ are of interest.

Suppose that f is analytic and has a zero at z_0, then for some $r > 0$ f has a Taylor expansion about z_0 on $B_r(b_0)$ as follows

$$\begin{aligned} f(z) &= c_m(z-z_0)^m + c_{m+1}(z-z_0)^{m+1} + \dots \\ &= (z-z_0)^m \sum_{k=0}^{\infty} c_{m+k}(z-z_0)^k \\ &= (z-z_0)^m g(z) \end{aligned}$$

where g is analytic on and has no zeros in $B_r(z_0)$. When f can be expressed in this way it is said to have a zero of order (or multiplicity) m at z_0.

Similarly, if f has a pole of order (or multiplicity) n at p_0 then it has a Laurent expansion on $B_r(p_0$ for some $r > 0$, say

$$\begin{aligned} f(z) = \frac{c_{-n}}{(z-p_0)^n} + \frac{c_{-n+1}}{(z-p_0)^{n-1}} + \dots + \frac{c_{-1}}{z-p_0} + c_0 + \dots \\ = \frac{1}{(z-p_0)^n} \sum_{k=0}^{\infty} c_{-n+k}(z-p_0)^k = \frac{g(z)}{(z-p_0)^n} \end{aligned}$$

where g is analytic on and has no zeros in $B_r(p_0)$.

Theorem 1 (Principle Of The Argument) *Let U be an open simply connected domain and γ be a positively oriented contour in U. If $f\colon U \to \mathbb{C}$ is analytic on U except at finitely many points (poles) and has finitely many zeros in U then*

$$\frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)} \, \mathrm{d}z = N - P$$

where f has N zeros and P poles in U.

Proof. Let $p_1, p_2, \dots, p_k$ be the poles of f inside γ, with the ith having multiplicity n_i. Let $z_1, z_2, \dots, z_l$ be the zeros of f inside γ, with the jth having multiplicity m_j. Choose $r > 0$ such that

$$\begin{aligned} f(z) &= (z-z_j) m_j g_j(z) \quad \forall z \in B_r(z_j) \\ \text{and } f(z) &= (z-p_i)^{-n_i} g_{l+i}(z) \quad \forall z \in B_r(p_i) \end{aligned}$$

where each g is non-zero and analytic on its respective ball. As there are only finitely many poles and zeros, r may be chosen small enough to make all the balls disjoint.

Take any j $(1 \leqslant j \leqslant k)$ then in $B_r(z_j)$

$$\begin{aligned}
f(z) &= (z - z_j)^{m_j} g_j(z_j) \\
\text{so } f'(z) &= m_j(z - z_j)^{m_j - 1} + (z - z_j)^{m_j} g_j'(z_j) \\
\text{giving } \frac{f'(z)}{f(z)} &= \frac{m_j}{z - z_j} + \frac{g_j'(z)}{g_j(z)} \\
&= \frac{m_j}{z - z_j} + h(z)
\end{aligned}$$

where h is analytic on $B_r(z_j)$. This function has a single pole at z_j with residue m_j. Similarly, take any i $(1 \leqslant i \leqslant l)$ then in $B_r(p_i)$

$$\begin{aligned}
f(z) &= (z - p_i)^{-n_i} g_{l+i}(z_i) \\
\text{so } f'(z) &= -n_i(z - p_i)^{-n_i - 1} + (z - p_i)^{-n_i} g_{l+i}'(z_i) \\
\text{giving } \frac{f'(z)}{f(z)} &= \frac{-n_i}{z - p_i} + \frac{g_{l+i}'(z)}{g_{l+i}(z)} \\
&= \frac{n_i}{z - p_i} + h(z)
\end{aligned}$$

where h is analytic on $B_r(p_i)$. This function has a single pole at p_i with residue $-n_i$. Now, poles of $\frac{f'(z)}{f(z)}$ can occur only at poles of f' (which must be poles of f) and zeros of f. Thus by Cauchy's Residue Formula

$$\frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)}\, \mathrm{d}z = \sum_{i=1}^{k} -n_i + \sum_{j=1}^{l} m_j = N - P$$

□

Corollary 2 $N - P = \dfrac{1}{2\pi i} \underset{gamma}{\triangle} \arg f(z)$

Proof.

$$\begin{aligned}
N - P &= \frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)}\, \mathrm{d}z \\
&= \frac{1}{2\pi i} \int_a^b \frac{f'(\gamma(t))}{f(\gamma(t))} \gamma'(t)\, \mathrm{d}t \\
&= \frac{1}{2\pi i} \left[\ln\left(f(\gamma(t))\right)\right]_a^b \\
&= \frac{1}{2\pi i} \left(\ln|f(\gamma(b))| + i \arg\left(f(\gamma(b))\right) - f(\gamma(a))| + i \arg\left(f(\gamma(a))\right)\right) \\
&= \frac{1}{2\pi i} (\arg\left(f(\gamma(b))\right) - \arg\left(f(\gamma(a))\right)) \\
&= \frac{1}{2\pi i} \underset{\gamma}{\triangle} \arg f(z)
\end{aligned}$$

where $\triangle_\gamma \arg f(z)$ denotes the change around γ of the argument of $f(z)$. □

This corollary suggests why the theorem is called the "Principle Of The Argument".

Applying a root-counting argument in fact allows a proof of the Fundamental Theorem of Algebra, a proof normally in the realm of algebra. Firstly, a lemma.

Lemma 3 *If $h(z) = \frac{c_1}{z} + \frac{c_2}{z^2} + \cdots + \frac{c_n}{z^n}$ for constants c_i, then $\exists R > 0$ such that $|h(z)| < 1$ for $z > R$.*

Proof. Using first the triangle inequality,

$$\begin{aligned}|h(z)| &\leqslant \frac{|c_1|}{|z|} + \frac{|c_2|}{|z|^2} + \cdots + \frac{|c_n|}{|z|^n} \\ &\leqslant \max_{1\leqslant i\leqslant n}\{|c_i|\}\left(\frac{1}{|z|} + \frac{1}{|z|^2} + \cdots + \frac{1}{|z|^n}\right) \\ &\leqslant \max_{1\leqslant i\leqslant n}\{|c_i|\}\frac{n}{|z|}\end{aligned}$$

with the last line following when $|z| > 1$. Hence choosing $R > \max\{1, n\max\{|c_i|\}\}$ the result is obtained.□

Theorem 4 (Fundamental Theorem Of Algebra) *If $p(z) \in \mathbb{C}[z]$ is of order $n \geqslant 1$ then p has precisely n zeros in $\mathbb{C}$. (More simply: $\mathbb{C}$ is algebraically closed.)*

Proof. Let $p(z) = a_0 + a_1 z + \cdots + a_n z^n$ then p is entire. Write

$$\begin{aligned}p(z) &= a_n z^n(1 + f(z)) \\ \text{with } f(z) &= \frac{a_{n-1}}{a_n}\frac{1}{z} + \frac{a_{n-2}}{a_n}\frac{1}{z^2} + \cdots + \frac{a_0}{a_n}\frac{1}{z^n}\end{aligned}$$

By Lemma 3 $\exists R > 0$ such that $|z| > R \;\Rightarrow\; |f(z)| < 1$. But then $1 + f(z) \neq 0$ and certainly $z \neq 0$ so that p has no roots outside the contour $|z| = R$, and no roots on the contour either.

Now, $1 > |f(z)| = |(1 + f(z)) - 1|$ so $1 + f(z)$ lies in the unit circle centred at 1. Therefore $\frac{-\pi}{2} < \arg(1 + f(z)) < \frac{\pi}{2}$ and so

$$0 \leqslant \left| \underset{|z|=R}{\triangle} (1 + f(z)) \right| \leqslant \pi$$

Applying the Principle Of The Argument, this must be an integer multiple of 2π, and thus is zero. Hence

$$\begin{aligned}\underset{|z|=R}{\triangle} p(z) &= \underset{|z|=R}{\triangle} (\arg a_n z^n) + \underset{|z|=R}{\triangle} (\arg(1 + f(z))) \\ &= \underset{|z|=R}{\triangle} \arg a_n z^n \\ &= \underset{|z|=R}{\triangle} \arg a_n |z|^n e^{in\theta} \\ &= 2n\pi\end{aligned}$$

By the Principle Of The Argument p has n roots and poles inside $|z| = R$, but as p has no poles it has n zeros, all in $\mathbb{C}$. □

Zeros and poles can also be counted by looking at the behaviour of functions on a contour. Where N_f denotes the number of zeros of f and P_f denotes the number of poles of f:

Theorem 5 (Generalised Rouché) *Let U be a simply connected open domain and let γ be a positively oriented simply connected closed contour in U. Let f and g be analytic on U, except for finitely many poles inside γ. If $|g(z)| < |f(z)|$ for all z on γ then*

$$N_f - P_f = N_{f+g} - P_{f+g}$$

Proof. For functions f and g as described, consider

$$F = \frac{f(z)}{f(z) + g(z)}$$

Using the quotient rule

$$F' = \frac{(f+g)f' - (f+g)'f}{(f+g)^2} \quad \text{so} \quad \frac{F'}{F} = \frac{\frac{(f+g)f'-(f+g)'f}{(f+g)^2}}{\frac{f(z)}{f(z)+g(z)}} = \frac{f'}{f} - \frac{(f+g)'}{f+g}$$

By the Principle Of The Argument (Theorem 1)

$$(N_f - P_f) - (N_{f+g} - P_{f+g}) = \frac{1}{2\pi i}\int_\gamma \frac{F'}{F} \tag{6}$$

and thus it suffices to show that this integral is zero. Now, as $|g(z)| < |f(z)|$, $\frac{|g(z)|}{|f(z)|} < 1$ and so

$$F = \frac{f}{f+g} = \frac{1}{1+\frac{g}{f}}$$

If this is assumed real, then it cannot be negative and so it is safe to make a cut along the negative real axis to make an analytic branch of the logarithm. Thus

$$\int_\gamma \frac{F'}{F}\,\mathrm{d}z = [\ln F]_{\gamma(a)}^{\gamma(b)} = 0$$

(because logarithm has been made analytic), which follows from Cauchy-Goursat. Hence equation (6) gives the required result. □

Corollary 7 (Rouché's Theorem) *Let U be a simply connected open domain and let γ be a positively oriented simply connected closed contour in U. Let f and g be analytic on U. If $|g(z)| < |f(z)|$ for all z on γ then $N_f = N_{f+g}$.*

Proof. Obvious from equation (6). □

(30.3) Contour Integration

(30.3.1) Residues

Contour integration and its applications use Cauchy's Residue Theorem (Theorem 9) heavily. To this end it important to be able to calculate residues.

Definition 8 *Let f be analytic on $B_r(a)$, except possibly at a. If $\sum_{\mathbb{Z}} c_n(z-a)^n$ is the Laurent expansion of f about a then the residue of f at a is c_{-1}.*

The following methods are available for calculating residues:

1. If f has a simple pole at a then $\operatorname{Res}(f,a) = \lim_{z\to a}(z-a)f(z)$.
2. If $f(z) = \frac{g(z)}{h(z)}$ where g and h are analytic on $B_r(a)$, $g(a) \neq 0$, $h(a) = 0$, $h'(a) \neq 0$, and f has a simple pole at a then
$$\operatorname{Res}(f,a) = \frac{g(a)}{h'(a)}$$
3. If $f(a) = \frac{g(a)}{(z-a)^m}$ where g is analytic in $B_r(a)$ and $g(a) \neq 0$ then f has a pole of order m at a which has residue
$$\operatorname{Res}(f,a) = \frac{1}{(m-1)!}g^{(m-1)}(a)$$
4. Calculate the Laurent expansion of f to find the coefficient of $(z-a)^{-1}$.

5. Calculate the integral $c_{-1} = \dfrac{1}{2\pi i}\displaystyle\int_\gamma f(z)\,\mathrm{d}z$.

The most frequently applicable of these is, unfortunately, the rather tedious method 4. To this end it is worth noting the following expansions

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \qquad e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} \qquad \sin z = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!} \qquad \cos z = \sum_{z=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!}$$

A useful trick with these is as follows:

$$\begin{aligned}
\operatorname{cosec}(\pi z) &= \frac{1}{\sin(\pi z)} \\
&= \frac{1}{\pi z - \frac{(\pi z)^3}{3!} + \frac{(\pi z)^5}{5!} - \dots} \\
&= \frac{1}{\pi z}\frac{1}{1 - \frac{(\pi z)^2}{3!} + \frac{(\pi z)^4}{5!} - \dots} \\
&= \frac{1}{\pi z}\frac{1}{1 - \left(\frac{(\pi z)^2}{3!} - \frac{(\pi z)^4}{5!} + \dots\right)} \\
&= \frac{1}{\pi z}\sum_{n=0}^{\infty}\left(\frac{(\pi z)^2}{3!} - \frac{(\pi z)^4}{5!} + \dots\right)^n \\
&= \frac{1}{\pi z} + \frac{1}{\pi z}\left(\frac{(\pi z)^2}{3!} - \frac{(\pi z)^4}{5!} + \dots\right) + \frac{1}{\pi z}\left(\frac{(\pi z)^2}{3!} - \frac{(\pi z)^4}{5!} + \dots\right)^2 + \dots
\end{aligned}$$

from which the coefficient of z^{-1} can be calculated.

Contour integrals may easily be evaluated by use of the following theorem.

Theorem 9 (Cauchy's Residue Theorem) *Suppose that γ is a simple closed contour in a domain D, and let f be a complex function which is analytic on D except at finitely many points, $p_1, p_2, \dots, p_k$, all of which line in* $\operatorname{Int}\gamma$. *Then*

$$\int_\gamma f(z)\,\mathrm{d}z = 2\pi i \sum_{i=1}^{k} \operatorname{Res}(f, p_i)$$

There are a few common choices of contour, each useful for integrating certain kinds of function. These are now examined by means of example.

(30.3.2) Semi-Circular Contours

Semi-circular contours are useful for evaluating real improper integrals of functions that behave badly at certain points in $\mathbb{R}$.

Example 10 *Evaluate* $\displaystyle\int_0^\infty \frac{\ln x}{a^2 - x^2}\,\mathrm{d}x$ *($a > 0$) by integrating round a suitable complex contour.*

Proof. Solution Use the contour shown in Figure 2, where

1. γ_1 is a contour along the positive real axis from ε to R, so on γ_1 $z = x$.
2. γ_2 is a semicircular contour centred at the origin and of radius R, so on γ_2 $z = Re^{i\arg\theta}$ for $0 \leqslant \theta \leqslant \pi$.
3. γ_3 is a contour along the negative real axis from $-R$ to $-\varepsilon$. Here, $\arg z = \pi$ so that on γ_3 $z = xe^{i\pi}$.
4. γ_4 is a semicircular contour centred at the origin and of radius ε, so on γ_4 $z = \varepsilon e^{i\theta}$ for $0 \leqslant \theta \leqslant \pi$.

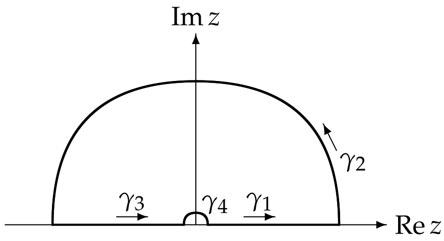

Figure 2: Contour of integration for use in evaluating improper real integrals with singularities.

Refer to the whole contour as $\Gamma_{R,\varepsilon}$ and consider the function $f(z) = \frac{\ln x}{a^2 - x^2}$. As a contour cannot cross a cut, define an analytic branch of the logarithm function by cutting from the plane the negative imaginary axis, so that $\arg z \in \left(\frac{-\pi}{2}, \frac{3\pi}{2}\right)$. The only pole lying inside the contour is at ia, and for it to lie inside the contour it is required that $\varepsilon < a < R$. This pole has residue

$$\operatorname{Res}(f, ia) = \frac{1}{2ia}\left(\ln a + i\frac{\pi}{2}\right)$$

Now go about evaluating $\int_{\Gamma_{R,\varepsilon}} f(z)\,\mathrm{d}z$ with the eventual aim of applying Theorem 9.

- On γ_1 $z = x$ so

$$\int_{\gamma_1} f(z)\,\mathrm{d}z = \int_{\varepsilon}^{R} \frac{\ln x}{a^2 + x^2}\,\mathrm{d}x = I_{R,\varepsilon}, \text{ say}$$

- On γ_3 $z = xe^{i\pi}$ so

$$\begin{aligned}
\int_{\gamma_3} f(z)\,\mathrm{d}z &= \int_{z=-R}^{z=-\varepsilon} \frac{\ln\left(xe^{i\pi}\right)}{a^2 - x^2 e^{2i\pi}} e^{i\pi}\,\mathrm{d}x \\
&= -\int_{x=R}^{x=\varepsilon} \frac{\ln\left(xe^{i\pi}\right)}{a^2 - x^2}\,\mathrm{d}x \\
&= \int_{\varepsilon}^{R} \frac{\ln\left(xe^{i\pi}\right)}{a^2 - x^2}\,\mathrm{d}x \\
&= \int_{\varepsilon}^{R} \frac{\ln x + i\pi}{a^2 - x^2}\,\mathrm{d}x \\
&= I_{R,\varepsilon} + \int_{\varepsilon}^{R} \frac{i\pi}{a^2 + x^2}\,\mathrm{d}x
\end{aligned}$$

- On γ_2 $z = Re^{i\theta}$ for $0 \leqslant \theta \leqslant \pi$ so

$$\begin{aligned}
\left|\int_{\gamma_2} f(z)\,\mathrm{d}z\right| &= \left|\int_0^{\pi} \frac{\ln\left(Re^{i\theta}\right)}{a^2 + R^2 e^{2i\theta}} Re^{i\theta}\,\mathrm{d}\theta\right| \\
&\leqslant \int_0^{\pi} \left|\frac{\ln\left(Re^{i\theta}\right)}{a^2 + R^2 e^{2i\theta}} Re^{i\theta}\right|\,\mathrm{d}\theta \\
&= \int_0^{\pi} \frac{|\ln R + i\theta||Re^{i\theta}|}{|a^2 + R^2 e^{2i\theta}|}\,\mathrm{d}\theta
\end{aligned}$$

Now, the denominator describes a circle centred at a^2 and of radius R^2. The minimum distance from the origin to this circle occurs when the circle crosses the real axis near the origin. Since $R^2 > a^2$ this gives

$$\begin{aligned}
&\leqslant \int_0^{\pi} \frac{|\ln R + i\theta||Re^{i\theta}|}{R^2 - a^2}\, \mathrm{d}\theta \\
&\leqslant \int_0^{\pi} \frac{(|\ln R| + \theta)R}{R^2 - a^2}\, \mathrm{d}\theta \quad \text{by the triangle inequality and since } R > 0 \\
&\leqslant \int_0^{\pi} \frac{(|\ln R| + \pi)R}{R^2 - a^2}\, \mathrm{d}\theta \\
&= \frac{\pi(|\ln R| + \pi)R}{R^2 - a^2} \\
&\to 0 \text{ as } R \to \infty
\end{aligned}$$

- On γ_4 $z = \varepsilon e^{i\theta}$ for $0 \leqslant \theta \leqslant \pi$ so

$$\begin{aligned}
\left|\int_{\gamma_4} f(z)\, \mathrm{d}z\right| &= \left|\int_{\pi}^{0} \frac{\ln\left(\varepsilon e^{i\theta}\right)}{a^2 + \varepsilon^2 e^{2i\theta}} \varepsilon e^{i\theta}\, \mathrm{d}\theta\right| \\
&\leqslant \int_0^{\pi} \left|\frac{\ln\left(\varepsilon e^{i\theta}\right)}{a^2 + \varepsilon^2 e^{2i\theta}} \varepsilon e^{i\theta}\right| \mathrm{d}\theta \\
&= \int_0^{\pi} \frac{|\ln \varepsilon + i\theta||\varepsilon e^{i\theta}|}{|a^2 + \varepsilon^2 e^{2i\theta}|}\, \mathrm{d}\theta
\end{aligned}$$

Now, the denominator describes a circle centred at a^2 and of radius ε^2. The minimum distance from the origin to this circle occurs when the circle crosses the real axis near the origin. Since $a^2 > \varepsilon^2$ this gives

$$\begin{aligned}
&\leqslant \int_0^{\pi} \frac{|\ln \varepsilon + i\theta||\varepsilon e^{i\theta}|}{a^2 - \varepsilon^2}\, \mathrm{d}\theta \\
&\leqslant \int_0^{\pi} \frac{(|\ln \varepsilon| + \theta)\varepsilon}{a^2 - \varepsilon^2}\, \mathrm{d}\theta \quad \text{by the triangle inequality and since } \varepsilon > 0 \\
&\leqslant \int_0^{\pi} \frac{(|\ln \varepsilon| + \pi)\varepsilon}{a^2 - \varepsilon^2}\, \mathrm{d}\theta \\
&= \frac{\pi(|\ln \varepsilon| + \pi)\varepsilon}{a^2 - \varepsilon^2} \\
&\to 0 \text{ as } \varepsilon \to 0
\end{aligned}$$

Hence by Theorem 9

$$2\pi i \frac{1}{2ia}\left(\ln a + i\frac{\pi}{2}\right) = 2\int_{\varepsilon}^{R} \frac{\ln x}{a^2 + x^2}\, \mathrm{d}x + i\int_{\varepsilon}^{R} \frac{\pi}{a^2 + x^2}\, \mathrm{d}x + \int_{\gamma_2} f(z)\, \mathrm{d}z + \int_{\gamma_4} f(z)\, \mathrm{d}z$$

Now, the left hand side is constant for all ε and R, thus so must the right hand side be. Hence taking the limit as $\varepsilon \to 0$ and $R \to \infty$ this must converge. In particular the real and imaginary parts converge, and so by equating real and imaginary parts

$$\int_0^{\infty} \frac{\ln x}{a^2 + x^2}\, \mathrm{d}x = \frac{\pi}{2a} \ln a \qquad \text{and} \qquad \int_0^{\infty} \frac{1}{a^2 + x^2}\, \mathrm{d}x = \frac{\pi}{a} \qquad \square$$

(30.3.3) Keyhole Contours

A keyhole contour is used to contain nearly all of the complex plane, with the 'gap' to make a cut.

Example 11 *Let $a \in \mathbb{C}$ with $0 < \operatorname{Re} a < 4$ and $a \neq 1,2,3$. Evaluate* $\displaystyle\int_0^\infty \frac{x^{a-1}}{x^4+1}\,\mathrm{d}x$.

Proof. Solution Consider $f(z) = \frac{z^{a-1}}{z^4+1}$ and take an analytic branch of 'powers' by cutting the complex plane along the positive real axis, so

$$z^{a-1} = e^{(a-1)\ln z} \qquad \ln z = \ln|z| + i\arg z \qquad \arg z \in (0, 2\pi)$$

f has 4 simple poles, each at a 4th root of -1, *i.e.*, $\omega_k = e^{i(2k+1)\frac{\pi}{4}}$ for $k = 0,1,2,3$.

$$\begin{aligned}
\operatorname{Res}(f, \omega_k) &= \left.\frac{z^{a-1}}{\frac{\mathrm{d}}{\mathrm{d}z} z^4 - 1}\right|_{z=\omega_k} \\
&= \left.\left(\frac{1}{4} z^{a-1} z^{-3}\right)\right|_{z=\omega_k} \\
&= \left.\left(\frac{1}{4} z^{a} z^{-4}\right)\right|_{z=\omega_k} \\
&= \frac{-1}{4}\omega_k^a
\end{aligned}$$

On γ_1, $z = x$ so

$$\int_{\gamma_1} f(z)\,\mathrm{d}z = \int_\varepsilon^R \frac{x^{a-1}}{x^4+1}\,\mathrm{d}x = I_{R,\varepsilon}, \text{ say}$$

On γ_3, $z = xe^{2\pi i}$, so

$$\int_{\gamma_3} f(z)\,\mathrm{d}z = \int_R^\varepsilon \frac{\left(xe^{2\pi i}\right)^{a-1}}{x^4 e^{8\pi i} + 1} e^{2\pi i}\,\mathrm{d}x = -\int_\varepsilon^R \frac{x^{a-1}e^{2\pi i a}}{x^4+1}\,\mathrm{d}x = -e^{2\pi i a} I_{R,\varepsilon}$$

Note that as $a \notin \mathbb{Z}$ the value of $e^{2\pi i a}$ is not known. For example if $a = \frac{1}{2}$ then the value of $e^{2\pi i a}$ could be 1 or -1.

On γ_2, $z = Re^{i\theta}$ for $0 \leqslant \theta \leqslant 2\pi$ so

$$\begin{aligned}
\left|\int_{\gamma_2} f(z)\,\mathrm{d}z\right| &= \left|\int_0^{2\pi} \frac{\left(Re^{i\theta}\right)^{a-1}}{R^4 e^{4i\theta} + 1} iRe^{i\theta}\,\mathrm{d}\theta\right| \\
&\leqslant \int_0^{2\pi} \frac{R\left|e^{(a-1)\ln R} e^{(a-1)i\theta}\right|}{R^4+1}\,\mathrm{d}\theta \\
&= \int_0^{2\pi} \frac{Re^{(\operatorname{Re}(a)-1)\ln R}\left|e^{i\operatorname{Im}(a)\ln R}\right| e^{-\operatorname{Im}(a)\theta}\left|e^{i\theta(\operatorname{Re}(a)-1)}\right|}{R^4+1}\,\mathrm{d}\theta \\
&= \int_0^{2\pi} \frac{RR^{\operatorname{Re}(a)-1}e^{-\operatorname{Im}(a)\theta}}{R^4+1}\,\mathrm{d}\theta \\
&= \frac{R^{\operatorname{Re}(a)}}{R^4-1}\int_0^{2\pi} e^{-\operatorname{Im}(a)\theta}\,\mathrm{d}\theta \\
&\to 0 \text{ as } R \to \infty \text{ because } \operatorname{Re}(a) < 4
\end{aligned}$$

Similarly $\int_{\gamma_4} f(z)\,\mathrm{d}z \to 0$ as $\varepsilon \to 0$. Hence by Theorem 9

$$\frac{-2\pi i}{4}\sum_{i=1}^{4}\omega_i = \left(1 - e^{2\pi i a}\right) I_{R,\varepsilon}$$

from which the solution follows. □

(30.4) Infinite Series

Contour integration can also be used to sum infinite series of the form $\sum_{n=0}^{\infty}\phi(n)$ and $\sum_{n=0}^{\infty}(\ =\ 1)^n\phi(n)$ where ϕ is a rational function.

Lemma 12 *Let ϕ be a rational function with poles $z_1, z_2, \ldots, z_k$. Then*

1. *$f(z) = \pi\cot(\pi z)\phi(z)$ has simple poles at each $n \in \mathbb{Z}$ and $n \notin \{z_1, z_2, \ldots, z_k\}$, with residue $\phi(n)$.*
2. *$f(z) = \pi\,\mathrm{cosec}\,(\pi z)\phi(z)$ has simple poles at each $n \in \mathbb{Z}$ and $n \notin \{z_1, z_2, \ldots, z_k\}$, with residue $(-1)^n\phi(n)$.*

Proof. Let

$$\begin{aligned} f(z) &= \pi\cot(\pi z) \\ &= \frac{\pi\cos(\pi z)\phi(z)}{\sin(\pi z)} \end{aligned}$$

then f has a simple pole for all $n \in \mathbb{Z}$ and $n \notin \{z_1, z_2, \ldots, z_k\}$. Calculating the residues,

$$\begin{aligned} \mathrm{Res}\,(f, n) &= \frac{\pi\cos(\pi n)\phi(n)}{\left(\frac{\mathrm{d}}{\mathrm{d}z}\sin(\pi z)\right)\Big|_{z=n}} \\ &= \phi(n) \end{aligned}$$

as required. Similarly for part (2). □

Lemma 13 *Let Γ_N be the square contour with vertices $\left(\pm(N+\frac{1}{2}), \pm i(N+\frac{1}{2})\right)$ for $N \in \mathbb{Z}$. Then both $\cot(\pi z)$ and $\mathrm{cosec}\,(\pi z)$ are bounded on Γ_N independently of N.*

Proof. Omitted. □

Using this,

$$\left|\int_{\Gamma_N}\pi\cot(\pi z)\phi(z)\,\mathrm{d}z\right| \leqslant M\left|\int_{\gamma_N}\phi(z)\,\mathrm{d}z\right| \times (\text{length of } \Gamma_N$$

and this will approach 0 if $\lim\limits_{z\to\infty} Z\phi(z) = 0$ so that by Theorem 9

$$0 = \sum_{\substack{n\in\mathbb{Z} \\ n\notin\{z_1,\ldots,z_k\}}}\phi(n) + \sum_{i=1}^{k}\mathrm{Res}\,(\pi\cot(\pi z)\phi(z), z_i)$$

where the residues at the z_i must each be calculated.

(30.5) Conformal Mappings

(30.5.1) Angle Preserving Maps

A conformal map is a function of $\mathbb{C} \to \mathbb{C}$ which, roughly speaking, preserves local structure. However, on a larger scale such a map can produce dramatic changes. For example, mapping a circle to the upper half plane.

Theorem 14 *Let G be an open subset of $\mathbb{C}$ and let $f\colon G \to \mathbb{C}$ be analytic on G. Let γ_1 and γ_2 be two contours in G that meet at a point z_0. If $f'(z_0) \neq 0$ then f preserves the magnitude and direction of the angles between γ_1 and γ_2 at z_0.*

Proof. By re-scaling the interval upon which γ_1 and γ_2 are defined it may be assumed that $\gamma_1(t_0) = z_0 = \gamma_2(t_0)$. Also, by rotating the plane if necessary, it may be assumed that $\gamma_2'(t_0) \neq 0$. Now, the angle between γ_1 and γ_2 at z_0 is the angle between the tangents. Hence

$$\text{angle between } \gamma_1 \text{ and } \gamma_2 \text{ at } z_0 = \arg(\gamma_1'(t_0)) - \arg(\gamma_2'(t_0)) = \arg\left(\frac{\gamma_1'(t_0)}{\gamma_2'(t_0)}\right) \tag{15}$$

Similarly, the angle between the images of γ_1 and γ_2 under f at z_0 is

$$\arg\left(\frac{(f\circ\gamma)_1'(t_0)}{(f\circ\gamma)_2'(t_0)}\right) = \arg\left(\frac{(f'(\gamma_1(t_0))\gamma_1'(t_0)}{(f'(\gamma_2(t_0))\gamma_2'(t_0)}\right)$$

by applying the chain rule. But

$$f'(\gamma_1(t_0) = f'(z_0) = f'(\gamma_2(t_0) \quad \text{and} \quad f'(z_0) \neq 0$$

hence cancelling, this is the same as equation (15), and so the result is shown. □

This motivates the following definition.

Definition 16 *Let G be an open subset of $\mathbb{C}$ and let $f\colon G \to \mathbb{C}$ be analytic on G. f is conformal at $z_0 \in G$ if $f'(z_0) \neq 0$, and conformal on G if $f'(z_0) \neq 0\ \forall z_0 \in G$.*

Some common and useful conformal maps are as follows.

- $f(z) = z + a$ for $a \in \mathbb{C}$. This is a translation in the direction $\overrightarrow{0a}$.
- $f(z) = ze^{i\alpha}$ for $\alpha \in \mathbb{R}$. This is an anticlockwise rotation through angle α about the origin.
- $f(z) = kz$ for $k \in \mathbb{R}^+$. This is a dilation centred at the origin.
- $f(z) = \frac{1}{z}$. This is called an inversion, and it exchanges the inside and outside of the unit circle.

(30.5.2) Straight Lines, Circles, And The Möbius Transformation

Theorem 17 *Let $\alpha, \beta \in \mathbb{C}$ with $\alpha \neq \beta$, and let $\lambda \in \mathbb{R}^+$. Then*

$$\left|\frac{z-\alpha}{z-\beta}\right| = \lambda$$

is the equation for a straight line or circle, and any straight line or circle has an equation of this form.

If $\lambda = 1$ then this is an equation for a straight line, otherwise the equation represents a circle.

Definition 18 *Let $a, b, c, d \in \mathbb{C}$ with $ad - bc \neq 0$ then the mapping*

$$f : z \mapsto \frac{az + b}{cz + d}$$

is called the Möbius transformation.

Clearly translations, rotations, dilations, and inversions are all kinds of Möbius transformation. All Möbius transformations are conformal maps, as

$$f'(z) = \frac{ad - bc}{(cz - d)^2}$$

Also, Möbius transformations are bijective, with inverse

$$f^{-1}(w) = \frac{b - dw}{cw - a}$$

Theorem 19 *The image under a Möbius transformation of a straight line or circle is either a straight line or a circle.*

Proof. Let L have equation $\frac{z-\alpha}{z-\beta} = \lambda$ for some $\alpha \neq \beta$ and $\lambda \in \mathbb{R}^+$. Let $f(z) = \frac{az+b}{cz+d}$ with $ad - bc \neq 0$ and let $w = f(z)$ then

$$z = \frac{dw - b}{a - cw}$$

substituting into the equation for L,

$$\begin{aligned} \lambda &= \left| \frac{\frac{dw-b}{a-cw} - \alpha}{\frac{dw-b}{a-cw} - \beta} \right| \\ &= \left| \frac{(dw - b) - \alpha(a - cw)}{(dw - b) - \beta(a - cw)} \right| \\ &= \left| \frac{(\alpha c + d)w - (\alpha a + b)}{(\beta c + d)w - (\beta a + b)} \right| \end{aligned} \tag{20}$$

which is in the form of a straight line or circle, or some degenerate case. There are 4 cases to consider.

- Suppose that $\alpha c + d = 0 = \beta c + d$. But $\alpha \neq \beta$, therefore $c = 0 = d$. But then $ad - bc = 0$, which contradicts that f is a Möbius transformation. Hence this case cannot occur.
- Suppose that $\alpha c + d \neq 0 \neq \beta c + d$ then dividing through in equation (20),

  $$\left| \frac{w - \frac{\alpha a+b}{\alpha c+d}}{w - \frac{\beta a+b}{\beta c+d}} \right| = \lambda \left| \frac{\beta c + d}{\alpha c + d} \right|$$
  $$\left| \frac{w - f(\alpha)}{w - f(\beta)} \right| = \lambda \left| \frac{\beta c + d}{\alpha c + d} \right|$$

  which is the equation of a straight line or circle.
- Suppose that $\alpha c + d \neq 0 = \beta c + d$ then equation (20) gives

  $$|w - f(\alpha)| = \left| \frac{\beta a + b}{\alpha c + d} \right|$$

  which is the equation of a circle.
- Suppose that $\alpha c + d = 0 \neq \beta c + d$ then equation (20) gives

  $$|w - f(\beta)| = \frac{1}{\lambda} \left| \frac{\alpha a + b}{\beta c + d} \right|$$

which is the equation of a circle. □

The next obvious question is as to precisely which straight lines and circles can be sent where. A straight line or circle is uniquely determined by three points so, as the next theorem shows, any straight line or circle an in fact be sent to any other.

Theorem 21 *For any pair of triples $\{z_1, z_2, z_3\}$ and $\{w_1, w_2, w_3\}$ with elements taken from $\mathbb{C} \cup \{\infty\}$ there is a unique Möbius transformation f for which $f(z_i) = w_1$ for each of $i = 1, 2, 3$.*

Proof. Choose $w_1 = 0$, $w_2 = 1$ and $w_3 = \infty$. This can be done without loss of generality because Möbius transformations are invertible, and so the transformation required by the theorem is then the composition $g^{-1} \circ f$ where

$$z_1 \xrightarrow{f} 0 \xleftarrow{g} w_1$$
$$z_2 \xrightarrow{f} 1 \xleftarrow{g} w_2$$
$$z_3 \xrightarrow{f} \infty \xleftarrow{g} w_3$$

Suppose then that $f(z) = \frac{az+b}{cz+d}$, then it is required that

$$\frac{az_1 + b}{cz_1 + d} = 0 \qquad \frac{az_2 + b}{cz_2 + d} = 1 \qquad \frac{az_3 + b}{cz_3 + d} = \infty$$

By the requirement from z_1, $b = -az_1$.
By the requirement from z_3, $d = -cz_3$.
Hence

$$f(z) = \frac{az - az_1}{bz - bz_3}$$

Now using the requirement from z_2,

$$1 = \frac{a}{c} \frac{z_2 - z_1}{z_2 - z_3}$$

which gives a value for $\frac{a}{c}$ and hence

$$f(z) = \left(\frac{z_2 - z_3}{z_2 - z_1}\right) \left(\frac{z - z_1}{z - z_3}\right)$$

Uniqueness follows since Möbius transformations are bijective. □

(30.5.3) Common Transformations

There are a number of common useful transformations. Moreover, these transformations can be combined to produce more exotic transformations.

- To map the unit circle to the right half plane use $f(z) = \frac{1+z}{1-z}$.
- To map the right half plane to the positive quarter plane first apply the power map $f(z) = \sqrt{z}$, then rotate through an angle of $\frac{\pi}{4}$ by using the map $g(z) = ze^{i\frac{\pi}{4}}$. The power map $f(z) = z^a$ for $a \in \mathbb{R}^+$ 'fans' the plane, either 'opening' or 'closing' it.
- The exponential map $f(z) = e^z$ has numerous uses.
  - To map the horizontal strip $\{z \mid a < \operatorname{Im} z < b\}$ to the wedge $\{z \mid a < \arg z < b\}$.
  - To map the vertical strip $\{z \mid a < \operatorname{Re} z < b\}$ to the annulus $\{z \mid e^a < |z| < e^b\}$.

When working with power maps it is very important to define an analytic branch of the logarithm. For example, $z \mapsto e^z$ will map $\{z \mid \operatorname{Re} z < a\}$ to the inside of the circle of radius e^a *but without the origin* because there is no room to make a cut in the plane without interfering with the set $\{z \mid \operatorname{Re} z < a\}$. Furthermore, this map cannot be inverted because once again, there is no room to make a cut in the plane in order to define an analytic branch of the logarithm.

Of course, if there is no readily available map, then it is possible to find an appropriate transformation using the method employed in the proof of Theorem 21.

(30.6) Analytic Functions

For a real valued function of a real variable, being differentiable limits somewhat the behaviour of a function. Similarly, for a complex valued function of a complex variable the condition of analycity admits certain behavioural properties of the function. Analycity is a far more strict condition than differentiability for real functions, and has some quite surprising consequences.

(30.6.1) Liouville's Theorem

Theorem 22 (Liouville) *If f is an entire function that is bounded, then f is constant.*

Proof. Choose any $a, b \in \mathbb{C}$ and let M be an upper bound for f on $\mathbb{C}$. Choose $R > \max\{|a|, |b|\}$ and let γ be a the contour $\{z \mid |z| = R\}$ then by Cauchy's Integral Formula

$$\begin{aligned}
|f(z) - f(b)| &= \left| \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z-a}\,\mathrm{d}z - \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z-b}\,\mathrm{d}z \right| \\
&\leqslant \frac{1}{2\pi} \sup_{z\in\gamma} \left| \frac{f(z)}{z-a} - \frac{f(z)}{z-b} \right| \times \text{length of } \gamma \quad \text{by the } ML\text{-result} \\
&= R \sup_{z\in\gamma} \left| \frac{(z-b)f(z) - (z-a)f(z)}{(z-a)(z-b)} \right| \\
&= R \sup_{z\in\gamma} \frac{|f(z)||a-b|}{|z-a||z-b|}
\end{aligned}$$

But $R > 2\max\{|a|, |b|\}$ and so for $z \in \gamma$, $|z-a| < \frac{R}{2}$ and $|z-b| < \frac{R}{2}$ hence

$$\begin{aligned}
&\leqslant R \frac{M|a-b|}{\left(\frac{R}{2}\right)^2} \\
&\to 0 \text{ as } R \to \infty
\end{aligned}$$

Note that R can be let tend to infinity as f is entire and hence this holds for all $R > 0$. □

The name of the next theorem is a little misleading. It is in fact completely equivalent to Liouville's Theorem: one can be deduced from the other.

Theorem 23 (Generalised Liouville) *Let f be entire and $k \in \mathbb{N}$. f is a polynomial of degree at most k if and only if $\exists M, K \in \mathbb{R}^+$ such that $|f(z)| \leqslant M|z|^k$ for $|z| \geqslant K$.*

Proof. ($\Rightarrow$) Suppose that f is a polynomial of degree at most k. Then

$$\begin{aligned}
f(z) &= a_k z^k + a_{k-1}z^{k-1} + \cdots + a_0 \\
\text{so } |f(z)| &\leqslant |a_k||z|^k + |a_{k-1}||z|^{k-1} + \cdots + |a_0| \quad \text{by the triangle inequality} \\
&\leqslant M'\left(|z|^k + |z|^{k-1} + \cdots + 1\right) \quad \text{where } M' = \max_{1\leqslant i\leqslant k} a_i \\
&\leqslant M'(k+1)|z|^k \quad \text{for } |z| > 1 \\
&= M|z|^k
\end{aligned}$$

where $M = M'(k+1)$.

($\Leftarrow$) Suppose that f is entire and that $|f(z)| \leqslant M|z|^k$ for $|z| \geqslant K$ where $M, K \in \mathbb{R}^+$ and $k \in \mathbb{N}$. Let $\gamma_R = \{z \mid |z| = R\}$ and $R \geqslant K$. Since f is entire it has a Taylor series expansion about any point $z \in \mathbb{C}$, so

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \quad \text{where} \quad a_n = \frac{1}{2\pi i}\int_{\gamma_R} \frac{f(z)}{z^{n+1}}\,\mathrm{d}z$$

Now, by the ML-result,

$$\begin{aligned}
|a_n| &\leqslant \frac{1}{2\pi}\sup_{z\in\gamma_R}\left|\frac{f(z)}{z^{n+1}}\right| \times \text{length of } \gamma_R \\
&\leqslant \frac{1}{2\pi}\frac{MR^k}{R^{k+1}}2\pi R \\
&\to 0 \text{ as } R \to \infty \text{ for } n > k
\end{aligned}$$

and this is valid for all $R > 0$ because f is entire. Hence f is a polynomial of finite degree of at most k. □

(30.6.2) The Identity Theorem

The Identity Theorem is a particularly powerful result regarding the structure of analytic functions. Its proof uses numerous analytical and topological results, which are now presented.

Definition 24 *Let f be a complex function. Define $\mathcal{Z}(f) = \{z \in \mathbb{C} \mid f(z) = 0\}$.*

Assertion 25 *Let $G \subseteq \mathbb{C}$ and $h\colon G \to \mathbb{C}$*

1. *If h is continuous then any limit point of $\mathcal{Z}(h)$ that is in G is in $\mathcal{Z}(h)$.*
2. *If G is open, h is continuous, and $h(a) \neq 0$ then $\exists r > 0$ such that $h(z) \neq 0$ for all $z \in B_r(a)$.*
3. *If G is open and A is a closed subset of G then $G \setminus A$ is open.*
4. *If G is open, $A \subseteq G$, and L is the set of limit points of A in G, then L is closed.*
5. *If G is open, A is a closed subset of G, and B is a closed subset of A, then B is a closed subset of G.*
6. *If G is open and connected, and A is a subset of G that is both closed and open in G, then either $A = G$ or $A = \emptyset$.*

Lemma 26 *Let G be an open subset of $\mathbb{C}$ and let $f\colon G \to \mathbb{C}$ be analytic. If $a \in \mathcal{Z}(f)$ then a is either an interior point or an isolated point.*

Proof. Take any $a \in \mathcal{Z}(f)$ then since f is analytic on G $\exists r > 0$ such that f has Taylor expansion

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n \quad \forall z \in B_r(a)$$

Since $f(a) = 0$ it is immediate that $a_0 = 0$. Further, either

1. $a_n = 0$ for all $n \in \mathbb{N} \cup \{0\}$, or
2. there must be some minimal m ($m > 0$ by the preceding statement) such that $a_m \neq 0$ but $a_j = 0$ for $j < m$.

In the first case $f(z) = 0$ for all $z \in B_r(a)$ and hence $B_r(a) \subseteq \mathcal{Z}(a)$, meaning that a is an interior point of $\mathcal{Z}(a)$.

In the second case,

$$f(a) = (z-a)^m \sum_{n=0}^{\infty} a_{n+m} (z-a)^n \quad \forall z \in B_r(a)$$

Let $g(z) = \sum_{n=0}^{\infty} a_{n+m}(z-a)^n$ then g is analytic on $B_r(a)$ and therefore is continuous. Also, $g(a) = a_m \neq 0$. But then $\exists s > 0$ with $s \leqslant r$ such that $g(z) \neq 0$ for all $z \in B_s(a) \setminus \{a\}$. But $f(z) = (z-a)^m g(z)$ which is then also non-zero on $B_s(a) \setminus \{a\}$ and hence $\mathcal{Z}(f) \cap B_s(a) = \{a\}$ *i.e.*, a is an isolated point of $\mathcal{Z} f$. □

Corollary 27 *If an analytic function f has an isolated zero at a then the zero is of finite multiplicity and $f(z) = (z-a)^m g(a)$ where $g(z) \neq 0$ for $z \in B_s(a)$ and g is analytic on G.*

Proof. As in case 2 in the proof of Lemma 26 define g by

$$g(z) = \begin{cases} a_m & \text{if } z = a \\ \frac{f(z)}{(z-a)^m} & \text{if } z \neq a \end{cases} \qquad \text{for } z \in B_s(a)$$

then g is analytic (including at a), and is non-zero in $B_s(a)$. □

Theorem 28 (Identity) *Let G be an open and connected subset of $\mathbb{C}$, and let $f\colon G \to \mathbb{C}$ be analytic. If $\mathcal{Z}(f)$ has a limit point in G, then f is identically zero.*

Proof. Let L be the set of limit points of $\mathcal{Z}(f)$ that lie in G.
By Assertion 25 $L \subseteq \mathcal{Z}(f)$ and is a closed subset of $\mathcal{Z}(f)$ which is a closed subset of G. Therefore L is a closed subset of G.

Take any $z_0 \in L$, then $z_0 \in \mathcal{Z}(f)$. Since $z_0 \in L$ it is not an isolated point of $\mathcal{Z}(f)$ and hence by Lemma 26 z_0 is an interior point of $\mathcal{Z}(f)$. But then $\exists r > 0$ such that $B_r(z_0) \subseteq \mathcal{Z}(f)$. Now, every point of $B_r(z_0)$ is a limit point of $B_r(z_0)$ and therefore $B_r(z_0) \subseteq L$, meaning that L is an open subset of G.

Hence L is both open and closed in G which is open and connected. Hence either $L = G$ or $L = \emptyset$.

Now, by hypothesis $\mathcal{Z}(f)$ has a limit point in G, therefore $L \neq \emptyset$. Hence $L = G$. But $L \subseteq \mathcal{Z}(f)$ and therefore $\mathcal{Z}(f) = G$, *i.e.*, f is identically zero on G. □