Chapter 35

Combinatorics

(35.1) Graphs

(35.1.1) Terms & De□itions

All of the following results can be found in Chapter **??**, and this is in no way a comprehensive summary of graph theory. The following definitions are as an *aide memoir*, ditto the results proven.

Definition 1 *A pseudograph Γ is a pair $\Gamma = (V, E)$ where V is a countable set and E is a family of un-ordered pairs of elements of V. Elements of V are called vertices, elements of E are called edges.*

- *If $v \in V$ and $\{v, v\} \in E$ then Γ has a loop.*
- *If not all elements of E are district, then Γ has parallel edges.*
- *A pseudograph with no loops or parallel edges is called a graph.*

Definition 2 *Let $\Gamma = (V, E)$ be a pseudograph. A sub-pseudograph of Γ is a pseudograph $\Gamma' = (V', E')$ where $V' \subseteq V$ and $E' \subseteq E'$ such that the elements of the pairs in E' are all elements of V'.*

Definition 3 *A walk in a graph $\Gamma = (V, E)$ is a sequence $v_1, v_2, \ldots, v_n$ of elements of V such that $\{v_i, v_{i+1}\} \in E$ for $1 \leqslant i < n$.*

- *If all the vertices are distinct then the walk is called a path.*
- *If all the edges are distinct then the walk is called a trail.*
- *An Euler trail is a trail which traverses all vertices exactly once, and $v_1 = v_n$.*

Theorem 4 *A connected graph $\Gamma = (V, E)$ contains an Euler trail if and only if all the vertex degrees are even.*

Proof. ($\Rightarrow$) Suppose that Γ has an Euler trail and consider tracing it. For any vertex v on this trail, each time v is visited an edge is 'used' on arrival, and a different edge on departure. Thus v must have even degree. The initial vertex is no exception since it is necessary to finish there.

($\Leftarrow$) By induction, if $|E| = 0$ then the result is trivial. Suppose that Γ has at least 1 edge, and that the result holds for graphs with fewer edges.

Since all vertex degrees are even Γ must contain a cycle $(v_0, v_1, \ldots, v_n)$ say. Let Γ' be the subgraph of Γ with all the edges of this cycle removed. Let $\gamma_1', \Gamma_2', \ldots, \Gamma_m'$ be the connected components of Γ', then by induction each has an Euler trail $\mathbf{v}_i$ say, and without loss of generality this may start and end at one of the vertices in the cycle in Γ. Hence an Euler trail in Γ is formed by tracing the cycle $(v_1, v_2, \ldots, v_n)$, and if v_i is a vertex in the Euler trail of Γ_j', then detour round that trail unless it has already been traced. $\square$

Definition 5 *Two pseudographs $\Gamma_1 = (V_1, E_1)$ and $\Gamma_2 = (V_2, E_2)$ are isomorphic if there exists a bijection $\phi\colon V_1 \to V_2$ such that*

$$\{v, w\} \in E_1 \;\Leftrightarrow\; \{\phi v, \phi(w)\} \in E_2$$

that is, ϕ preserves incidence in the edge set.

Note that since ϕ is a bijection, the isomorphism property defines a bijection $\psi\colon E_1 \to E_2$ such that $\psi(\{v, w\}) = \{\phi(v), \phi(w)\}$.

(35.1.2) Maps

Much of this chapter is concerned with colouring maps. Given a map, all its countries must be coloured in such a way that if two countries border eachother then they are different colours. Note that the sea counts as a country and must also be coloured.

Given a map a graph can be constructed whose edges correspond to boundaries, whose vertices correspond to the meeting of boundaries, and hence whose faces correspond to regions on the map.

Definition 6 *A pseudograph is a plane pseudograph if it can be drawn in the plane without any edges crossing. A pseudograph is planar if it is isomorphic to a plane pseudograph.*

In complete generality, a map is a pseudograph with an embedding in the plane. However, simplifications can be made by using a more distinct definition. Moreover these simplifications make no loss of generality.

- A vertex is only required where boundaries of different countries meet. Hence all vertex degrees may be assumed to be at least 3.
- If there is a loop then precisely 2 countries have a border at that loop. If the subgraph inside the loop is n-colourable and ditto the graph outside the loop, then by induction the whole graph is n colourable.

  For the inductive step observe that if the two countries that border at the loop are the same colour, then transposing this colour with another colour in the inside (say) of the loop results in an n-colouring of the map.

  Hence it may be assumed that there are no loops.
- If there are parallel edges, then as for a loop they divide the plane into 2 and the same argument applies. It may therefore be assumed that there are no parallel edges.
- If the pseudograph is disconnected, then if each component is n-colourable a suitable permutation of the colours shows that, by induction, the whole graph is n-colourable. Hence it may be assumed that the pseudograph is connected.
- Obviously, no country will have a boundary with itself. (One might think of such an edge as a county boundary, say.) Hence it can be assumed that the pseudograph does not have a bridge *i.e.,* an edge whose removal would disconnect the pseudograph.

Hence the following definition is made without any loss of generality.

Definition 7 *A map is a connected plane graph with no bridges and where all vertices have degree at least 3.*

Working with faces can be a little tricky—it is much easier to work with vertices. This motivates the following definition.

Definition 8 *Let $\Gamma = (V, E)$ be a graph. Define the dual of Γ to be the graph obtained from Γ by placing a vertex in each face of Γ and joining vertices which lie in adjacent faces.*

If Γ has p vertices, q edges, and r faces then its dual has r vertices, q edges, and p faces:

- If a vertex v is of degree d then it is surrounded by d faces. Hence in the dual v becomes a face with d sides.
- Every edge of Γ is a boundary between regions and so corresponds to an edge in the dual.
  - If an edge bounds distinct faces then it connects distinct vertices in the dual.
  - If an edge is a bridge then both 'sides' of it lie against the same face and so in the dual that edge becomes a loop.

So there is a 1-to-1 correspondence between the edges of Γ and its dual. Vertices and faces are in correspondence in which vertex degree and number of sides correspond accordingly.

(35.1.3) Euler's Formula

Euler's formula states quite simply that for a polyhedron with p vertices, q edges, and r faces that $p - q + r = 2$. This is in fact also true for plane pseudographs. Firstly a rather technical lemma is needed.

Lemma 9 *If a plane pseudograph has at least one edge, and no vertex has degree 1 then it contains a cycle.*

Proof. Let $e = \{v_1, v_2\}$ be an edge. If $v_1 = v_2$ then there is nothing more to show, so suppose this is not so. Now, as v_2 does not have degree 1 there is a vertex v_3 adjacent to v_2. Continuing in this way, any vertex v_i is adjacent to some other vertex v_{i+1} and this may continue indefinitely.

Since the pseudograph is finite $\exists i$ such that $v_{i+1} = v_j$ for some $j < i$ and hence $v_j, v_{j+1}, \ldots, v_{i+1}$ is a closed path. But these vertices are distinct and so this is the required cycle. □

Theorem 10 (Euler) *Let Γ be a connected plane pseudograph with p vertices, q edges, and r faces. Then $p - q + r = 2$.*

Proof. If Γ has a vertex of degree 1 then its removal decreases p by 1 and q be 1 thus leaving $p - q + r$ unchanged. It may therefore be assumed that Γ has no vertices of degree 1.

By Lemma 9 Γ contains a cycle. Let e be an edge of this cycle, then e can be removed from Γ leaving a connected graph. The removal of e reduces q by 1 and reduces r by 1 and so the value of $p - q + r$ is unchanged.

Now, if the resulting graph has any vertices of degree 1 then as above they can be removed. If it has no vertex of degree 1 then Lemma 9 applies and an edge can be removed. This paragraph then applies again.

As Γ is finite eventually all edges will be removed to leave just 1 vertex. But at no point has the value of $p - q + r$ changed, and for 1 vertex $p - q + r = q - 0 + 1 = 2$, which must therefore hold for Γ. □

Euler's formula can be used to establish a number of useful results connecting the numbers of vertices, edges, and faces. The 'handshaking lemma',

$$2|E| = \sum_{v \in V} \deg v$$

can also be deduced from Euler's formula, as is now shown.

Lemma 11 (Hand shaking) *If Γ is a pseudograph with p_i vertices of degree i and q edges then*

$$\sum_{i=1}^{\infty} i p_i = 2q$$

Proof. Observe that there are $2q$ half-edges and that each is incident with precisely 1 vertex. Furthermore, a vertex of degree i has i half-edges incident with it. $\square$

An alternative proof of this lemma is to count the elements of the adjacency matrix firstly by row and then by column, equating the results. This is a classic example of counting the same thing in two different ways.

For a pseudograph the hand shaking lemma can also be applied to its dual. Hence the following.

Lemma 12 *If Γ is a plane pseudograph with r_i faces with i edges then*

$$\sum_{i=1}^{\infty} i r_i = 2q$$

Proof. Consider walking the boundary of each face. Each edge is traversed once on each 'side' and so is counted twice. $\square$

Theorem 13 *In any planar graph Γ with at least three vertices $q \leqslant 3p - 6$.*

Proof. The proof may be split into cases depending on Γ.

1. If Γ has a vertex of degree 1 then its removal creates a graph with $p-1$ vertices and $q-1$ edges. Hence
$$q - 1 \leqslant 3(p-1) - 6 \quad \text{so} \quad q \leqslant 3p - 3 - 6 + 1 \leqslant 3p - 6$$
Hence it is now only necessary to consider graphs with no vertex of degree 1.
2. If Γ is connected and has no bridges or vertices of degree 1 then each edge bounds 2 distinct faces. By definition, each face has at least 3 edges and so Lemma 12 becomes $\sum_{i=3}^{\infty} i r_i = 2q$. Therefore
$$\begin{aligned} \sum_{i=3}^{\infty} 3r_i = 3r &\leqslant 2q \\ 3p - 3q + 3r &= 6 \\ 3p - 3q + 2q &\geqslant 6 \\ q &\leqslant 3p - 6 \end{aligned}$$
3. If Γ is connected and has a bridge but has no vertex of degree 1 then the bridge connects subgraphs Γ_1 and Γ_2. Both of these have at most 1 vertex of degree 1: the one from which the bridge was removed. Hence each has at least 3 vertices and so by induction these satisfy
$$q_1 \leqslant 3p_1 - 6 \qquad q_2 \leqslant 3p_2 - 6$$
Hence for Γ,
$$q = q_1 + q_2 + 1 \leqslant 3p - 12 + 1 \leqslant 3p - 6$$
To start the induction consider a graph with 3 vertices. $3p - 6 = 3$ and such a graph can have at most 3 edges, so the proof is complete.
4. If Γ is disconnected and has no vertex of degree 1 then each connected component must have at least 3 vertices and thus by induction for component i, $q_i \leqslant 3p_i - 6$. Summing over all connected components gives the result.

All cases are covered, so the theorem is proven. □

Note this theorem does not preclude the existence of non-plane graphs with $q \leqslant 3p-6$. The contrapositive form of the statement shows that if this inequality does not hold then the graph is not planar.

Definition 14 *The girth of a graph is the length of the shortest cycle.*

Theorem 15 *Let Γ be a plane graph of girth l. Then*

$$q \leqslant \frac{l}{l-2}(p-2)$$

Proof. The proof is by induction on the number of edges in a graph of girth l. The smallest graph of girth l is simply a cycle of length l, which consists of l vertices and l edges. In this case

$$\frac{l}{l-2}(p-2) = \frac{l}{l-2}(l-2) = l = q$$

and so the result holds.

1. Suppose that Γ has a vertex of degree 1. If this vertex and its edge are removed then the resulting graph has $p-1$ vertices and $p-1$ edges and is of girth l. Hence by induction

$$\begin{aligned} q-1 &\leqslant \frac{l}{l-2}(p-3) \\ q &\leqslant \frac{l}{l-2}(p-2) + \left(1 - \frac{l}{l-2}\right) \end{aligned}$$

   But $\frac{l}{l-2} < 1$ because l is at least 3 and hence the result holds.

2. Suppose that Γ is connected, has no vertex of degree 1, and has a bridge. Let Γ_1 and Γ_2 be the subgraphs of Γ that are connected by the bridge.

   Any cycle in Γ cannot cross the bridge for if it did then it would have to cross again, contradicting the uniqueness of vertices in a cycle. Hence any cycle in Γ lies in Γ_1 or Γ_2. Hence at least one of these components must be of girth l (Γ_1 say) and the other must either be of girth at least l or not have any cycles.

   In Γ, no vertex of Γ_2 has degree 1 and therefore there is a cycle in Γ_2. By the above paragraph this cycle remains when the bridge is removed, and this cycle must be of length at least l, length $l+k$ say. Now,

$$\begin{aligned} \frac{l+k}{l+k-2} - \frac{l}{l-2} &= \frac{(l+k)(l-2) - l(k+k-2)}{(l+k-2)(l-2)} \\ &= \frac{-2k}{(l+k-2)(l-2)} \end{aligned}$$

   But $l \geqslant 3$ and $k \geqslant 1$, therefore this is negative. Therefore

$$q_2 \leqslant \frac{l+k}{l+k-2}(p_2-2) \quad \Rightarrow \quad q_2 \leqslant \frac{l}{l-2}(p_2-2)$$

   By induction

$$q_1 \leqslant \frac{l}{l-2}(p_1-2)$$

and hence

$$\begin{aligned} q &= q_1 + q_2 + 1 \\ &\leqslant \frac{l}{l-2}(p_1 - 2) + q_2 \\ &\leqslant \frac{l}{l-2}(p_2 - 2) + 1 \\ &= \frac{l}{l-2}(p - 2) + \left(1 - \frac{2l}{l-2}\right) \end{aligned}$$

But $\frac{2l}{l-2} \geqslant \frac{2l}{l} = 2$ and therefore the last term is negative. Hence the result holds.

3. Suppose that Γ is disconnected and has no vertex of degree 1. It is now necessary to induce on the number of connected components. The case of 1 connected component is covered in the other cases. The induction step is essentially the same as for 2 connected components. The argument proceeds as above—with one component having girth at least l—though this time the sum $q_1 + q_2$ is examined instead of $q_1 + q_2 + 1$.
4. Suppose that Γ is connected, has no vertex of degree 1, and has no bridges. Hence by Lemma 12 $2q \geqslant lr$ and hence using $r \geqslant \frac{2q}{l}$ in Euler's formula

$$p - 2 = q - r \quad \text{so} \quad p - 2 \geqslant q - \frac{2q}{l} = \frac{q(l-2)}{l}$$

which gives the result. □

Continuing in this vein, some other results about plane graphs are available.

Theorem 16 *If Γ is a plane pseudograph with all vertices of degree at least 3 then Γ has a face with at most 5 sides.*

Proof. Suppose that all faces have at least 6 sides, then by the Hand shaking lemmas,

$$\begin{aligned} 2q &= \sum_{i=6}^{\infty} ir_i \\ &\geqslant 6r \end{aligned} \qquad\qquad \begin{aligned} 2q &= \sum_{i=3}^{\infty} ip_i \\ &\geqslant 3p \end{aligned}$$

Hence in Euler's formula,

$$p - q + r \leqslant \frac{2}{3}q - q + \frac{2}{6}q = 0$$

which is a contradiction. □

(35.2) Colouring

(35.2.1) The Four Colour Problem

Given any plane graph how many colours are needed to colour the map so that adjacent countries have different colours? This question has been pondered since the mid nineteenth century, though was not solved until the 1970s, requiring the aid of a computer.

Definition 17 *A Kempe chain containing a face A in a plane pseudograph is a maximal sequence of faces of alternating colours of which A is a member.*

For example, choose any face A of colour red, say, and then a red-blue Kempe chain is obtained by including any blue face adjacent to A, then any red face adjacent to any of the blue faces...

Note that the colours of a Kempe chain can be swapped. For example in a red-blue chain all the red regions can be re-coloured blue and vice versa.

Lemma 18 *Let Γ be a 4-colourable map. If 4 regions meet at a vertex v then the map can be re-coloured so that only 3 colours are used for these 4 regions.*

Proof. Say the regions are coloured red, green, blue, and yellow in this order around v. Consider a red-green Kempe chain containing the red region.

- If this Kempe chain does not include the green region then the red-green Kempe chain can be re-coloured by switching red and green, leaving only 3 colours around v.
- If tis Kempe chain does include the green region, consider a blue-yellow chain that includes the blue region.
  - If the blue-yellow Kempe chain does not include the yellow region then its colours can be swapped, leaving only 3 colours around v.
  - If the blue-yellow Kempe chain connects to the yellow region then it must cross the red-green chain, which is impossible.

In any case the map can be suitably re-coloured, so the result is shown. □

Lemma 19 *Let Γ be a 4-colourable map. If 5 regions meet at a vertex v then the map can be re-coloured so that only 3 colours are used for these 5 regions.*

Proof. Fallacious Proof Suppose the colouring around v is red, green, blue, yellow, green in this order.

- If a red-blue Kempe chain does not connect the red region with the blue region, then the map can be suitably recoloured.
- If a red-blue Kempe chain does connect the red region with the blue region then consider a red-yellow Kempe chain.
- The remaining case is when the red region is connected to both the blue region and the yellow region by Kempe chains. In this case both green regions are isolated by a red-$\substack{\text{blue}\\\text{yellow}}$ Kempe chain.
  - For the green region isolated by the red-blue Kempe chain consider a green-yellow Kempe chain. This may be re-coloured.
  - For the green region isolated by the red-yellow Kempe chain consider a green-blue Kempe chain. This may be re-coloured.

Hence in any case the graph can be suitably re-coloured. □

From this result Kempe proved the 4-colour theorem: this proof is fallacious. This is illustrated in Figure 1. The counter example is a little intricate, indeed the proof was believed for 10 years.

(35.2.2) The Five Colour Theorem

Although Kempe's proof was false, Kempe chain arguments are still extremely useful, and some similar results can be salvaged.

Theorem 20 *Every map can be coloured with at most six colours.*

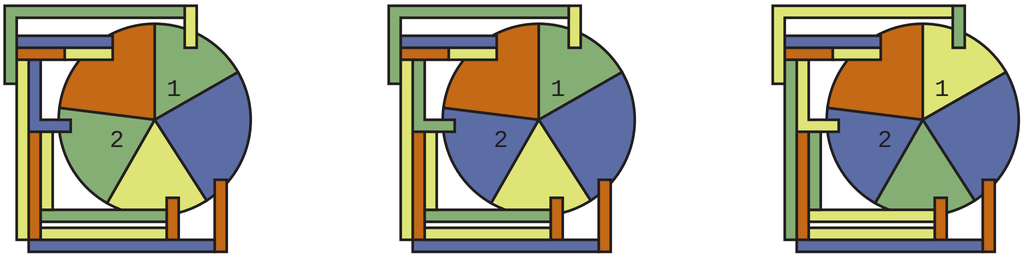

Figure 1: Counter example to Kempe's proof of the 4-colour theorem.
Re-colouring the green_2-blue chain creates a green_1-yellow chain, so the green_1 region cannot be re-coloured.

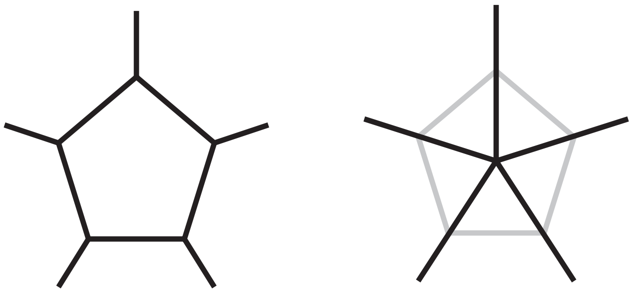

Figure 2: A region is divided amongst its neighbours.

Proof. By Theorem 16 such a map contains a face with 5 or fewer sides. Divide this region amongst its five neighbours, so that a new vertex in the centre of the region has the five neighbouring countries round it (see Figure 2). By induction the resulting map can be coloured with at most six colours. Reinstating the removed region, it was surrounded by at most 5 other faces and so a colour is available for it. □

With the aid of a lemma, this can be improved.

Lemma 21 *Let Γ be a 5-colourable map. If 5 regions meet at a vertex v then the map can be re-coloured so that only 4 colours are used for these 5 regions.*

Proof. Suppose that the colours are, in order, red, orange, yellow, green, and blue. If the red and yellow regions are not in the same Kempe chain then the red region can be re-coloured as yellow. If they are, then there can be no orange-blue Kempe chain and so the orange region can be re-coloured blue. □

Theorem 22 (Heawood) *Every plane map can be coloured with at most 5 colours.*

Proof. By Theorem 16 such a graph contains a face with 5 or fewer sides which can be divided amongst its neighbours as per Figure 2. By induction the resulting map can be 5-coloured.

When the removed region is reinstated, if it has 4 or fewer neighbours then there is a colour available for it. If it has 5 neighbours then Lemma 21 permits a suitable re-colouring. □

(35.2.3) Reduction

A general pseudograph may have many peculiarities. It has already been shown that only plane maps need be considered in proving the 4-colour theorem. Further simplifications are available.

Theorem 23 *If the 4-colour theorem holds for cubic plane maps then it holds for all plane maps.*

Note that a cubic or trivalent graph is a graph where every vertex has degree 3.

Proof. Wherever more than 3 regions meet introduce a new region at that vertex which has 1 border with each of the originally meeting regions. This results in a cubic graph, and if the new graph can be 4-coloured then so can the old one since if regions are not adjacent in the new graph then they are not adjacent in the old graph. □

Theorem 24 *A minimal counter example to the 4-colour theorem must contain at least 12 faces that have 5 sides.*

Proof. **** □

(35.2.4) The Three Colour Theorem

Using the correspondence of faces and vertices of a graph and its dual, some results are easier to work with. Three-face-colouring is equivalent to 3-vertex-colouring in the dual, which motivates the following theorem.

Theorem 25 *Suppose that Γ is a plane graph and all its faces have 3 sides. Γ is 3-vertex-colourable if and only if Γ has an Euler circuit.*

Proof. ($\Rightarrow$) Consider the dual of Γ, which must be a cubic map any by hypothesis is 3-face-colourable. Suppose there exists a face A with oddly many sides. Colouring alternately the faces surrounding A at least 3 colours are needed. But then A must be coloured a fourth colour; a contradiction. Hence every face in the dual of Γ has evenly many edges, hence every vertex in Γ has even degree and so by Theorem 4 Γ has an Euler trail.

($\Leftarrow$) By Theorem 4 all vertices of Γ have even degree, so all faces of the dual of Γ, $\overline{\Gamma}$ say, have evenly many edges. By Theorem 16 $\overline{\Gamma}$ has a face with at most 5 sides. But all faces have evenly many sides and no face can have 2 sides, therefore there exists a face with exactly 4 sides.

Colour this face red, then the surrounding faces can be coloured green and blue. Merge the red face with its 2 green neighbours to create one large green country. By induction the resulting map can be 3-face-coloured, and therefore so $\overline{\Gamma}$. Hence Γ can be 3-vertex-coloured. □

Corollary 26 *A plane cubic graph is 3-face-colourable if and only if every face has evenly many edges.*

Proof. Translation of Theorem 25 into dual form. □

The 4-colour theorem mat also be translated into dual form: it then says that a planar graph is 4-vertex-colourable. This form is considerably easier to formulate.

(35.2.5) Hamilton Cycles

Unlike Euler trails, Hamilton cycles are particularly difficult to find.

Theorem 27 *If a cubic map has a Hamilton cycle then it can be 4-face-coloured.*

The Hamilton cycle has an inside and outside both consisting of a chain of faces. Each may be coloured alternately, thus using 4 colours. This theorem can be strengthened.

Theorem 28 *If a map has a Hamilton cycle then it can be 4-face-coloured.*

Proof. Consider the dual graph induced on the faces on the inside (outside) of the Hamilton cycle.

- If there is a cycle then the enclosed face corresponds to a vertex in the original graph that is not part of the Hamilton cycle; a contradiction.
- If there is more than 1 connected component then the original graph contains a cut-vertex which contradicts the existence of a Hamilton cycle.

Hence the graph induced on the faces of the inside (outside) of the Hamilton cycle is a tree and thus can be 2-coloured. This corresponds to a 2-colouring of the faces which lie on the inside (outside) of the Hamilton cycle which thus gives a 4-colouring of the map. □

These theorems are due to Tait in 1830. He believed that all cubic graphs contained a Hamilton cycle and thus he had proven the 4-colour theorem. However, in 1896 Petersen found a cubic graph that does not contain a Hamilton cycle: the Petersen graph. The Petersen graph is not planar, and a counter example for maps was not found until 1946.

(35.2.6) Edge Colouring

A graph can be k-edge-coloured if the edges can be coloured in such a way that no two edges that are incident with the same vertex are assigned the same colour. Clearly for a graph Γ at least $\Delta(\Gamma)$, the maximum vertex degree, colours are required.

Theorem 29 *If Γ is a cubic graph with a Hamilton cycle then Γ can be 3-edge-coloured.*

Proof. Since Γ is cubic $2q = 3p$ and therefore p must be even. Therefore the Hamilton cycle has evenly many edges and so these edges can be 2-coloured. Since Γ is cubic vertex edge has exactly 1 remaining edge and this can be coloured with a third colour. □

Theorem 30 (Petersen) *Let Γ be a cubic graph. Γ is 3-edge-colourable if and only if the vertex set of Γ is spanned by a family of disjoint cycles all of even length.*

Proof. ($\Rightarrow$) If Γ can be 3-edge coloured then choose any two colours. The edges of these colours forms a collection of disjoint cycles of even length that spans the vertex set of Γ.

($\Leftarrow$) Each of the disjoint cycles of even length can be 2-coloured with the same 2 colours. Any 2 of the remaining edges can never be incident with the same vertex as Γ is cubic, and so they can all be coloured with the same third colour. □

Theorem 31 (Petersen-Tait) *Let Γ be a cubic plane graph that does not have a bridge. Γ can be 3-edge-coloured if and only if Γ can be 4-face-coloured.*

Proof. ($\Rightarrow$) Suppose that Γ can be 3-edge-coloured. Choose any 2 of the 3 colours then since all 3 colours of edge are incident at any vertex (since there is no bridge) a cycle can be traced that consists of edges of the chosen 2 colours.

For any choice of 2 colours these cycles are disjoint and span the vertex set of Γ. Any face is contained within a number of such cycles. Say the edge colours are red, green, and blue then the faces may be classified and coloured as given in Table 1.

It must now be shown that no two faces of the same colour, as assigned in Table 1, meet.

- If adjacent faces meet at a green edge then the number of red-green cycles in which they lie differs by 1 and the number of red-blue cycles in which they lie is the same. From Table 1 these faces have different colours.

Face colour:	orange	yellow	purple	black
Number of red-green cycles enclosing the face:	odd	odd	even	even
Number of red-blue cycles enclosing the face:	odd	even	odd	even

Table 1: Classification of faces according to enclosure by cycles.

- If adjacent faces meet at a blue edge then the number of red-blue cycles in which they lie differs by 1 and the number of red-green cycles in which they lie is the same. From Table 1 these faces have different colours.
- If adjacent faces meet at a red edge then the number of red-green cycles in which they lie differs by 1, ditto the number of red-blue cycles. From Table 1 these faces have different colours.

($\Leftarrow$) Suppose that Γ can be 4-face-coloured. Now, $\binom{4}{2} = 6$ and therefore the edges fall into 6 categories, determined by colours of the faces the edge bounds. Since Γ is cubic, any two edges that meet at a particular vertex border a common face. Hence

- An orange-yellow edge cannot meet a purple-black edge. If it did then the vertex at which they meet would have degree at least 4. These edges can all be coloured red.
- An orange-purple edge cannot meet a yellow-black edge. These edges can all be coloured green.
- An orange-black edge cannot beet a yellow-purple edge. These edges can all be coloured blue.

Hence a 3-edge-colouring has been constructed. □

Note that if a graph is 4-colourable then it may be assumed that there are no bridges. Furthermore, in a cubic graph with a bridge applying the Hand Shaking lemma for each side of the bridge, $3p_1 = 2q_1$ and $3p_2 = 2q_2$. If the bridge is removed then

$$3(p_1 - 1) + 2 = 2(q_1 - 1) \qquad\qquad 3(p_2 - 1) + 2 = 2(q_2 - 1)$$
$$3(p_1 - 1) = 2q_1 - 4 \qquad\qquad 3(p_2 - 1) = 2q_2 - 4$$

But in both cases the right hand side is even, therefore each side of the bridge must have oddly many vertices. No cycle can cross the bridge and so it is impossible for the vertex set to be spanned by cycles of even length. Hence a cubic graph can have no bridge.

The condition of having no bridge may therefore be omitted from Theorem 31.

As for face and vertex colouring, a number of results concerning the exact number of colours required are available.

Definition 32 *Let Γ be a graph. Define the (edge-)chromatic index of Γ, $\chi'(\Gamma)$, to be the minimum number of colours required to edge-colour Γ.*

As already noted, $\chi'(\Gamma) \geqslant \Delta(\Gamma)$. Now, each edge is adjacent to at most $2\Delta(\Gamma) - 1$ different edges: when the edge is incident with two vertices of degree $\Delta(\Gamma)$. Thus $2\Delta(\Gamma) - 2$ colours will always be sufficient.

Theorem 33 (König) *If Γ is a bipartite graph then $\chi'(\Gamma) = \Delta(\Gamma)$.*

Proof. If Γ has two vertices and one edge then the result is trivial. Suppose the result holds for bipartite graphs with fewer then $n > 1$ edges and consider a graph with n edges. Remove an edge $\{u, v\}$ from Γ to create a subgraph Γ' which, by induction, has $\chi'(\Gamma') = \Delta(\Gamma')$.

In Γ' the vertices u and v both have degree at most $\Delta(\Gamma) - 1$. Furthermore, $\Delta(\Gamma) \geqslant \Delta(\Gamma')$ and therefore in Γ there is a colour not used at u, and there is a colour not used at v.

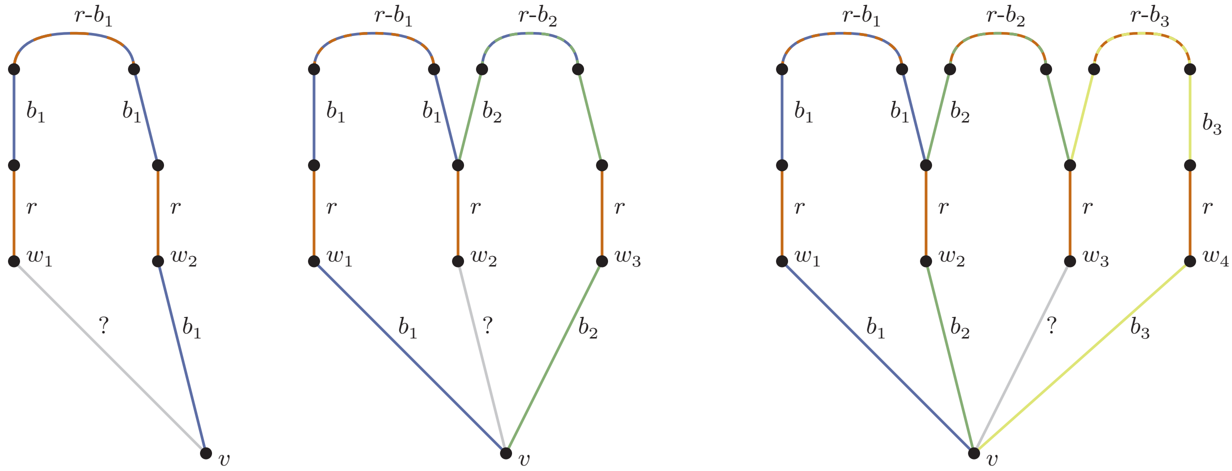

Figure 3: Re-colouring edges in the inductive step of Vizing's Theorem.

If these colours are the same the result follows trivially. For the remaining case suppose that only red is missing at u and only blue is missing at v. Consider the red-blue Kempe chain from v: Since Γ is bipartite any uv path must be of odd length and thus if the red-blue Kempe chain contains u then it arrives along a red edge. But there is no red edge at u, so this chain can be re-coloured thus eliminating red from v. The edge $\{u, v\}$ may then be coloured red. $\square$

The number of colours required to edge-colour a graph is in fact very tightly bounded.

Theorem 34 (Vizing) *For any graph Γ, $\chi'(\Gamma) \leqslant \Delta(\Gamma) + 1$.*

Proof. By induction on the number of edges, removing an edge $\{v, w_1\}$ from Γ leaves a graph that can by edge-coloured with at most $\Delta(\Gamma) + 1$ colours.

If there is a colour that is not used at either v or w_1 then $\{v, w_1\}$ can be coloured in Γ with this colour.

If there is not a colour that is absent from both v and w_1, say red (r) is absent from v but not w_1, and blue (b_1) is absent from w_1 but not v. Consider the r-b_1 Kempe chain from v and including w_2, as shown in Figure 3. If the chain does not include w_1 then it can be re-coloured and then $\{v, w_1\}$ can be coloured blue.

The remaining case is when the r-b_1 Kempe chain from v including w_2 includes w_1. Colour $\{v, w_1\}$ as b_1 and consider colouring $\{v, w_2\}$ a different colour. From previously r is absent from v, and the problem is when some colour b_2 of an edge $\{v, w_3\}$ is absent from w_2 but has a r-b_2 Kempe chain including w_2: Figure 3.

Once again, re-colour $\{v, w_2\}$ as b_2 and consider the problem of colouring $\{v, w_3\}$.

This process continues, and since there are finitely many colours there will be a vertex w_k at which the missing colour is b_j for $j < k$ *i.e.*, a colour that has already been 'processed': see Figure 4.

Since b_j is absent from w_k the r-b_j Kempe chain through w_j and w_{j+1} cannot contain w_k. Hence a r-b_1 chain from w_k can be re-coloured, allowing $\{v, w_k\}$ to be coloured red. $\square$

Hence graphs may be categorised as either "Class 1" where $\chi'(G) = Delta(G)$, or as "Class 2" where $\chi'(G) = \Delta(G) + 1$. Most graphs turn out to be Class 1; Class 2 maps are difficult to find.

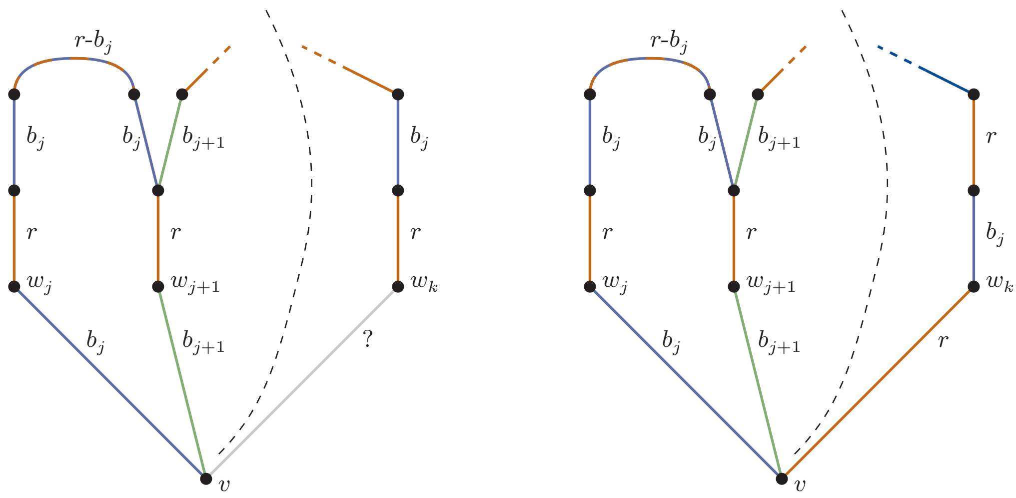

Figure 4: Re-colouring a Kempe chain in the inductive step of Vizing's Theorem.

(35.3) Maps On Different Surfaces

Thus far only maps in an infinite 2-dimensional space have been considered: This is topologically equivalent to sphere*, which is a solid with one surface that has no 'holes'.

(35.3.1) Surfaces With Holes

A torus differs from a sphere in that it has a "hole". Consider a map on the surface of a torus; the "hole" can be used to join vertices that could not otherwise be joined on the surface of a sphere.

Of course, a surface may have more than one "hole":

Definition 35 *A smooth closed orientable (has an "inside" and an "outside") surface with g "holes" is said to be of genus g.*

Definition 36 *A map of genus g is a map which may be drawn on a surface of genus g but no surface of lesser genus.*

For example, K_7 may be drawn on the surface of a torus so that it is a map, thus K_7 is a map of genus 1.

There is a subtle problem with maps on surfaces of non-zero genus: They aren't really maps unless each face can be drawn in the plane, which such a surface cannot. It is required that each face can be "flattened out" into a plane in such a way that the surface is not cut in that region.

Theorem 37 (Euler) *For a map drawn on a surface of genus g that has p vertices, q edges, and r faces, $p - q + 4 = 2(1 - g)$.*

(35.3.2) The Surface Of A Torus

Consider cutting a torus and bending it round to form a cylinder that has one circular end identified with the other. The cylinder may then be cut along its length and flattened out into a rectangle, such that opposite edges are identified.

*The Riemann Sphere

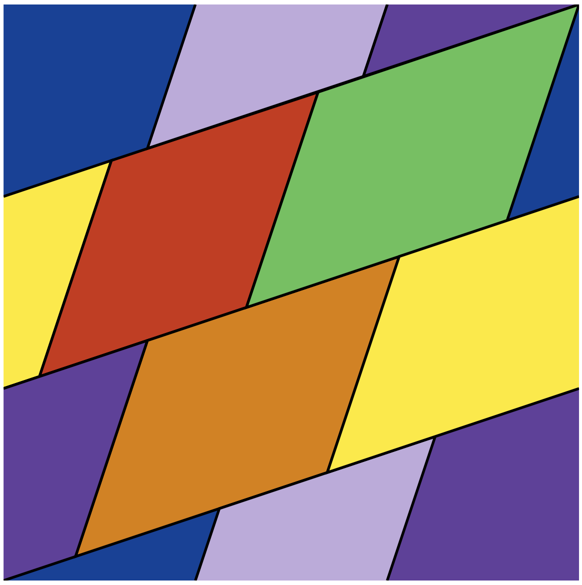

Figure 5: 2-dimensional diagram of a map on the surface of a torus that requires 7 colours.

Just as on the surface of a sphere, it may be assumed that all maps are cubic, so that by the Handshaking Lemma $2q = 3p$. Hence by the generalised form of Euler's Formula (Theorem 37),

$$0 = 3p - 3q + 3r = 3r - q \quad \Rightarrow \quad q = 3r$$

Now, if the average number of neighbours to a face is d then by the Handshaking Lemma $2q = dr$ and hence $d = 6$ and so there is a face that has at most 6 neighbours.

Theorem 38 *Every map on the surface of a torus can be face-coloured with at most 7 colours.*

Proof. If the map has 7 or fewer faces then the result is trivial. For a map with at least 8 faces there is a face with at most 6 neighbours. Remove an edge from this face to give a map with fewer faces which, by induction, can be 7-coloured. Replacing this edge, the neighbours of the face with at most 6 sides have been coloured with at most 6 colours, leaving a seventh colour for this face. □

In fact there exists a map that requires 7 colours: it consists of 7 mutually adjacent hexagons, and is shown in Figure 5 and in Figure 6. Hence 7 colours are necessary and sufficient. Notice that this result is considerably easier than the 4-colour problem.

Just as with maps on the surface of a sphere the dual may be formed, and the following equivalent to Theorem 38 is then available.

Theorem 39 *The vertices of any map drawn on a surface of genus 1 can be coloured with at most 7 colours.*

Proof. If there are 7 or fewer vertices the result is trivial. If there are 8 or more vertices since it may be assumed that there are no multiple edges, each face has at least 3 sides. Hence by the Handshaking Lemma, $3r \leqslant 2q$. Hence

$$0 = 3p - 3q + r3 \leqslant 3p - q \quad \Rightarrow \quad q \leqslant 3p$$

Now, if d is the average vertex degree then by the Handshaking Lemma $2q = dp$ and hence $d \leqslant 6$. Hence there exists a vertex v with degree at most 6. Remove v and all edges incident with it, then by induction

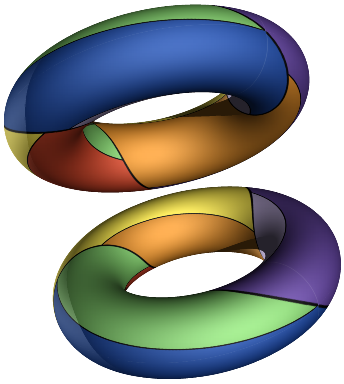

Figure 6: 3-dimensional picture of a map on the surface of a torus that requires 7 colours.

the resulting graph can be 7-vertex-coloured. Replacing v, since it ahs only at most 6 neighbours there is a seventh colour available for it. □

(35.3.3) Surfaces Of Higher Genus

Euler's Formula has already been generalised. As on the surface of a torus, it is not difficult to calculate how many colours are needed to colour the faces of a map on a surface of genus $g \geqslant 2$.

Theorem 40 *For a map of genus g, $q \leqslant 3p + 6g - 6$.*

Proof. Since all faces have at least 3 sides, the Handshaking Lemma gives $2q \geqslant 3r$. Hence in Euler's formula,

$$6(1-g) = 3p - 3q + 3r \leqslant 3p - q$$ □

Theorem 41 *For a graph of genus g and girth l,*

$$q \leqslant \frac{l}{l-2}(p + 2g - 2)$$

Proof. For a graph of girth l the Handshaking Lemma gives $2q \geqslant lr$ so that in Euler's formula,

$$\begin{aligned} p - q + r &= 2(1-g) \\ p - q + \frac{2q}{l} &\geqslant 2(1-g) \\ p - \frac{q(l-2)}{l} &\geqslant 2(1-g) \\ q &\leqslant \frac{l}{l-2}(p - 2 + 2g) \end{aligned}$$ □

Alternatively, this result may be used to give a lower bound on the genus of surface on which a particular map can be drawn, *i.e.*,

$$g \geqslant 1 + \frac{1}{2}\left(\frac{q(l-2)}{l} - p\right)$$

Lemma 42 *If Γ is a graph of genus g and d is the average vertex degree then*

$$d = 6 + \frac{12g - 12}{p}$$

Proof. Since all faces are triangles, by the Handshaking Lemma $2q = 3r$ and $2q = dp$. Hence in Euler's formula,

$$\begin{aligned} 6(1-g) &= 3p - 3q + 3r \\ &= 3p - q \\ &= 3p - \frac{dp}{2} \\ d &= 6 + \frac{12g-12}{p} \end{aligned}$$

□

Note that in general $2q \geqslant 3p$ and so

$$d \leqslant 6 + \frac{12g-12}{p}$$

Theorem 43 *If Γ is a map of genus $g \geqslant 2$ then it is k-colourable where*

$$k = \lfloor \frac{7+\sqrt{1+48g}}{2} \rfloor$$

Proof. The maximum vertex degree is at most $p-1$ and hence by Lemma 42

$$\begin{aligned} 6 + \frac{12g-12}{p} &\leqslant p-1 \\ 6p + 12g - 12 &\geqslant p^2 - p \\ p^2 - 7p - 12g + 12 &\geqslant 0 \end{aligned}$$

Now, the roots of this can be easily calculated. One of them is negative, and hence

$$p \geqslant h = \frac{7+\sqrt{1+48g}}{2}$$

Now, $h^2 - 7h + 12 - 12g = 0$ and hence

$$h - 7 = \frac{12g-12}{h}$$

so that by Lemma 42 again

$$\begin{aligned} d &= 6 + \frac{12g-12}{p} \\ &\leqslant 6 + \frac{12g-12}{h} \\ &= 6 + (h-7) \\ &= h-1 \end{aligned}$$

Hence there must be a vertex of degree at most $k-1$ where $k = \lfloor h \rfloor$. Now proceeding by induction on the number of vertices, remove a vertex of degree at most $k-1$ and triangulate the graph. By induction the resulting graph can be $(k-1)$-coloured so that Γ can be k-coloured. □

(35.4) Non-planar Graphs

(35.4.1) Connectivity

Definition 44 *The connectivity of a graph Γ, $\kappa(\Gamma)$, is the minimum number of vertices that must be removed in order to disconnect the graph or reduce it to a single vertex.*

The clause about reduction to a single vertex arrises due to complete graphs, where every vertex is connected to every other vertex.

Definition 45 *A graph Γ is k-connected if $k \leqslant \kappa(\Gamma)$,*

Clearly the connected components of a graph are the maximal 1-connected subgraphs. Indeed, a graph is 1-connected if and only if it is connected.

Definition 46 *The blocks of a graph Γ are the maximal 2-connected subgraphs of Γ, and bridges with their 2 vertices.*

Definition 47 *A graph Γ is n-chromatic if n is the minimum number of colours required to colour the vertices of Γ such that no two adjacent vertices are coloured in the same colour. Write $\chi(\Gamma) = n$.*

Note that being n-chromatic is different to being n-colourable, since if a graph Γ is n-colourable then $\chi(\Gamma) \leqslant n$.

The proper generalisation of the 4-colour theorem is to determine the chromatic index of an arbitrary graph: in the case of planar maps this is the 4-colour problem in dual form.

Certainly a graph with maximum vertex degree Δ can be coloured with $\Delta + 1$ colours: Each vertex is adjacent to at most Δ other vertices. In fact a better bound is available.

Theorem 48 (Brooks) *If Γ is a connected graph with maximum vertex degree Δ then Γ can be Δ vertex coloured, unless Γ is isomorphic to $K_{\Delta+1}$ or $\Delta = 2$ and Γ has a cycle of odd length.*

Proof. Firstly, note that if Δ is isomorphic to $K_{\Delta+1}$ then $\Delta + 1$ colours are needed. If $\Delta = 2$ and Γ has a cycle of odd length then clearly 3 colours are needed.

It may be assumed that Γ has no cut-vertex, since if it does then the two subgraphs which are joined at the cut-vertex can be Δ coloured by induction, and the colourings matched up at the cut-vertex.

It may also be assumed that every vertex has degree Δ, since if a vertex v has degree less than Δ then it can be removed, the resulting graph coloured with Δ colours (by induction), and then replaced. Since v is adjacent to less than Δ colours there is a colour available for it.

Suppose that Γ is 3-connected, has n vertices, and is not isomorphic to K_n. Since the labelling of vertices is arbitrary, choose n_n to be adjacent to v_1 and v_2 which are not adjacent.

Now choose v_i so that it is adjacent to at least one of $v_{i+1}, v_{i+2}, \ldots, v_n$. If this cannot be done, then there is a vertex v that is not adjacent to any of $v_n, v_{n-1}, \ldots, v_i$ and so removing v_1 and v_2 from Γ results in a disconnected graph. This contradicts the fact that Γ is 3-connected.

Now to colour the graph. v_1 and v_2 can be coloured with the same colour. By construction v_i is adjacent to at least one vertex of greater index, and so can be adjacent to at most $\Delta - 1$ of the vertices $v_1, v_2, \ldots, v_{i-1}$ and hence it can be coloured with one of Δ colours. v_n is adjacent to v_1 and v_2 which have the same colour, and is also adjacent to at most $\Delta - 2$ other vertices and thus can also be coloured with one of Δ colours.

Suppose Γ is not 3-connected, then from above it may be assumed that Γ is 2-connected, so let u, v_n be a pair of vertices which disconnects Γ. Consider removing v_n and all edges incident to it to give a graph Γ'.

Since Γ' is 1-connected and not 2-connected, it has 2 end-blocks that would be joined in Γ by v_n. Choose v_1 in one of these two blocks such that v_1 is adjacent to n_n, and similarly, choose v_2 in the other block and adjacent to v_n. Furthermore, since $\Delta \geqslant 3$ v_1 and v_2 can be chosen so that neither of them with v_n disconnects Γ.

Now apply the same colouring algorithm as before. □

Definition 49 *Let Γ be a graph. Define the vertex independence number $\alpha(\Gamma)$ to be the maximum number of non-adjacent vertices* i.e., *the maximum number of vertices which can be coloured with the same colour.*

Theorem 50 $\dfrac{p}{\alpha\Gamma} \leqslant \chi(\Gamma) \leqslant p - \alpha(\Gamma) + 1.$

Proof. If Γ is vertex coloured, then any set of vertices of the same colour must be mutually non-adjacent. If every such set has the maximum possible size, $\alpha(\Gamma)$ then $\frac{p}{\alpha(\Gamma)}$ colours are needed. Clearly in general more colours are needed.

Choose a set of $\alpha(\Gamma)$ mutually non-adjacent vertices, then the remaining vertices can be coloured with at most $p - \alpha(\Gamma)$ colours. Hence the result. □

Definition 51 *Let $\Gamma = (V, E)$ be a graph. The complement of Γ, $\overline{\Gamma}$, is a graph with vertex set V and such that $u, v \in V$ are adjacent if and only if they are not adjacent on Γ.*

Trivially, if $\Gamma = (V, E)$ and $\overline{\Gamma} = (V, \overline{E})$ then the graph $(V, E \cup \overline{E})$ is the complete graph on V.

Intuitively, if $\chi(\Gamma)$ is small, then there can't be many edges. But then $\overline{\Gamma}$ has lots of edges and so $\chi(\overline{\Gamma})$ should be large. This is formalised as follows.

Theorem 52 (Nordhaus & Gaddum) *Let Γ be a graph with p vertices. Then*

$$2\sqrt{p} \leqslant \chi(\Gamma) + \chi(\overline{\Gamma}) \leqslant p + 1 \quad \textit{and} \quad p \leqslant \chi(\Gamma)\chi(\overline{\Gamma}) \leqslant \left(\frac{p+1}{2}\right)^2$$

Proof. Γ has a set of $\alpha(\Gamma)$ mutually non-adjacent vertices and therefore $\overline{\Gamma}$ has as a subgraph the complete graph on these vertices so that $\chi(\overline{\Gamma}) \geqslant \alpha(\Gamma)$. Hence

$$\chi(\Gamma)\chi(\overline{\Gamma}) \&geq \chi(\Gamma)\alpha(\Gamma)$$
$$\geqslant p \quad \text{by Theorem 50}$$

Let v be a vertex of Γ and let Γ' be the graph obtained from Γ by removing v. Then $\overline{\Gamma'}$ is the graph obtained from $\overline{\Gamma}$ by removing v. By induction

$$\chi(\Gamma') + \chi(\overline{\Gamma'}) \leqslant (p-1) + 1 = p$$

Let v have degree d, then in $\overline{\Gamma}$ the degree of v is $p - 1 - d$.

- If $d \leqslant \chi\Gamma'$ then Γ' can be coloured with $\chi(\Gamma')$ colours. Since v is adjacent to fewer than $\chi(\Gamma')$ vertices, it can be coloured in Γ with one of $\chi(\Gamma')$ colours so $\chi(\Gamma) = \chi(\Gamma')$. Trivially $\chi(\overline{\Gamma}) \leqslant \chi(\overline{\Gamma'}) + 1$ and hence

$$\chi(\Gamma) + \chi(\Gamma') \leqslant \chi(\Gamma') + \chi(\overline{\Gamma'}) + 1 \leqslant p + 1$$

- If $p - 1 - d \leqslant \chi(\overline{\Gamma'})$ then the same argument applies in the dual graphs.

- If neither of these cases hold then

$$\begin{aligned} d &\geqslant \chi(\Gamma') \\ p-1-d &\geqslant \chi(\overline{\Gamma'}) \\ (\chi(\Gamma')+1)+(\chi(\overline{\Gamma'})+1) &\leqslant d+(p+d-1)+2 \\ \chi(\Gamma)+\chi(\overline{\Gamma}) &\leqslant p-1 \end{aligned}$$

For the remaining two inequalities, observe that for positive numbers a and b the arithmetic mean is greater than or equal to the geometric mean, *i.e.*,

$$\frac{a+b}{2} \geqslant \sqrt{ab}$$

Using this gives

$$\begin{aligned} \frac{\chi(\Gamma)+\chi(\overline{\Gamma})}{2} &\geqslant \sqrt{\chi(\Gamma)\chi(\overline{\Gamma})} \\ \frac{\chi(\Gamma)+\chi(\overline{\Gamma})}{2} &\geqslant \sqrt{p} \end{aligned} \qquad\qquad \begin{aligned} \frac{\chi(\Gamma)+\chi(\overline{\Gamma})}{2} &\geqslant \sqrt{\chi(\Gamma)\chi(\overline{\Gamma})} \\ \frac{p+1}{2} &\geqslant \sqrt{\chi(\Gamma)\chi(\overline{\Gamma})} \end{aligned}$$

which completes the proof. □

However, this result does not say much about the number of edges: this information is locked in the relationship between a graph and its complement. The following result is a little more satisfying.,

Theorem 53 $\chi(\Gamma) \geqslant \dfrac{p^2}{p^2-2q}$.

Proof. Suppose that Γ is k-coloured, and let $p_1, p_2, \ldots, p_k$ be the number of vertices of each colour. No two vertices of the same colour are adjacent, and therefore they are adjacent in $\overline{\Gamma}$ so that if $\overline{\Gamma}$ has n vertices then

$$\begin{aligned} n &\geqslant \binom{p_1}{2} + \binom{p_2}{2} + \cdots + \binom{p_k}{2} \\ &= \sum_{i=1}^{k} \frac{p_i}{p_i - 1} 2 \\ &= \frac{1}{2}\sum_{i=1}^{k} p_i^2 - \frac{1}{2}\sum_{i=1}^{k} p_i \\ &\geqslant \frac{1}{2k}\left(\sum_{i=1}^{k} p_i\right)^2 - \frac{1}{2}\sum_{i=1}^{k} p_i \end{aligned}$$

where the last line follows from the Cauchy-Schwartz inequality,

$$\langle \mathbf{u}, \mathbf{v} \rangle \leqslant \|\mathbf{u}\| \|\mathbf{v}\|$$

Now, $n+q$ is the number of edges in the complete graph on p vertices. Noting that $\sum_{i=1}^{k} p_i = p$ this gives

$$\begin{aligned} n+q = \binom{p}{2} &= \frac{p(p-1)}{2} \\ \frac{p(p-1)}{2} - q &\geqslant \frac{p^2}{2k} - \frac{p}{2} \\ p^2 - p - 2q &\geqslant \frac{p^2}{k} - p \\ k &\geqslant \frac{p^2}{p^2-2q} \end{aligned}$$

□

(35.4.2) Hadwiger's Conjecture

As already mentioned, in a general setting it is of interest to find the chromatic index of a graph. In contrapositive form the 4-colour conjecture may be re-stated as "every 5-chromatic graph is non-planar".

Definition 54 *An elementary contraction of a graph is obtained by identifying two vertices v and w, which may be replaced by a single vertex x.*

A contraction of a graph is obtained by a sequence of elementary contractions.

Conjecture 55 (Hadwiger) *Every n-chromatic graph has a subgraph which contracts to K_n.*

The case $n = 4$ was proved by Hadwiger when this conjecture appeared in 1943, and is proven below. The cases $n \geqslant 7$ are still un-proven.

Theorem 56 *If a graph Γ is 4-chromatic then it has a subgraph which contracts to K_4.*

In contrapositive form, if Γ has no subgraph which contracts to K_4 then Γ is not 4-chromatic. Since the minimum number of colours required to colour Γ is not 4, Γ must be 3-colourable: this is now proven.

Proof. By induction, it may be assumed that Γ is connected, and since trees are 2-colourable it may be assumed that Γ has a cycle. Let $v_1, v_2, \ldots, v_k$ be a cycle of minimal length and let $\Gamma_1', \Gamma_2', \ldots, \Gamma_r'$ be the connected components of the graph obtained by removing the vertices $v_1, v_2, \ldots, v_k$ from Γ.

If Γ_i is adjacent to at least 3 vertices of the cycle, then contracting the cycle to these 3 vertices and contracting Γ_i to a single vertex produces K_4: a contradiction. Hence each Γ_i is adjacent to at most 2 vertices of the cycle. (See Figure 7.) If Γ_i is adjacent to only 1 vertex in the cycle, then this vertex is a cut vertex and so by induction Γ can be 3-coloured. Hence each Γ_i may be assumed to be adjacent to precisely 2 vertices of the cycle.

Call these two vertices the feet of Γ_i and the adjoining edges the legs. Now, if any legs cross (see Figure 7) then the 4 feet contract to K_4: a contradiction. Therefore the Γ_i' occur in 'nests'.

If $r = 1$ then there is a vertex v of the cycle that is not adjacent to any vertex of Γ_1'. By induction the graph obtained from Γ by removing v can be 3-coloured. But v is adjacent to only 2 other vertices, so the theorem is proved.

If $r > 1$ then there are at least 2 nests: see Figure 8. Let v and w be vertices of the cycle which are the outermost feet of nests, then Γ may be split onto 2 subgraphs, Γ_1 and γ_2, the intersection of which is the graph $(\{v, w\}, \varnothing)$. To each of these subgraph add the edge $\{v, w\}$ (if it is not already there) then by induction Γ_1 and Γ_2 can be 3-coloured and a permutation of the colours will allow the colours of v and w in Γ_1 and Γ_2 to match, thus producing a 3-colouring of Γ. □

(35.5) The 4-colour Theorem

(35.5.1) Strategy For Proof

The main idea of the proof is not greatly dissimilar to that of Kempe in his original 'proof': See Lemma 19. The general inductive step is to remove a vertex v, 4-colour the resulting graph (by induction), and then find a way to colour v. The vertex v is said to be reducible.

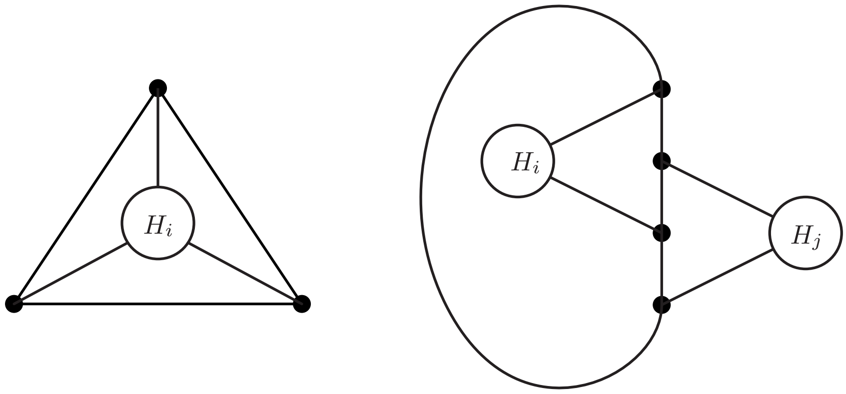

Figure 7: Contractions to K_4 in the proof of Hadwiger's Theorem.

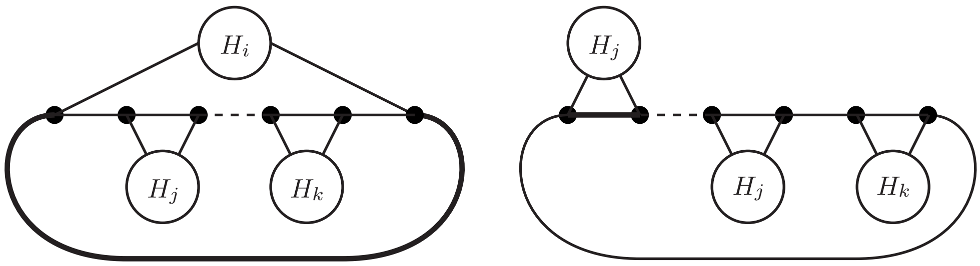

Figure 8: Proof of Hadwiger's Theorem: There are at least 2 nests.
The thick vertices correspond: Simply re-arranging the graph shows that if there is only 1 nest, then in fact there are at least 2.

Trivially, if v has degree 3 then it can be coloured. Lemma 18 provides a method to colour v if it had degree 4. Hence it has already been shown that graphs that have a vertex of degree at most 4 are reducible. The problem now is to reduce graphs which have no vertex of degree less than 5.

By Theorem 16 every planar graph has a vertex of degree 5, and without loss of generality only triangulated graphs need be considered.

To prove the 4-colour theorem it is therefore necessary and sufficient to find a reducible vertex in an arbitrary triangulated graph with no vertex of degree less than 4.

Definition 57 *A configuration is a cycle (called the bounding ring) with a specified interior graph.*

- *A configuration is reducible if it contains a reducible vertex.*
- *A set of configuration is unavoidable if every planar graph contains at least one of them.*

The task, then is to identify an unavoidable set, and show that each configuration is reducible. Originally 1,936 unavoidable configurations were identified, though this has since been reduced to just 633. Nevertheless, the use of a computer to check reducibility is advisable.

(35.5.2) Reducibility

Reduction

A reduction theorem is of the form "if a graph contains a certain configuration then it is reducible". Besides this kind of result, more information is available about what properties a counter-example to the four colour theorem must have.

Lemma 58 *A planar graph Γ is maximal planar if and only if it is a triangulation.*

Proof. ($\Rightarrow$) Suppose Γ is maximal planar and that some region has 4 or more sides, then it has 4 or more vertices on its boundary. If all these vertices are mutually adjacent (on the exterior of the region) then add a new vertex to the centre of the region and join the boundary vertices to it. This creates a complete graph on at least 5 vertices which cannot be planar and hence at least 2 or the boundary vertices are not adjacent. These can be joined on the interior of the region, contradicting maximal planarity.

($\Leftarrow$) If Γ is a triangulation then $3r = 2q$ and hence by Theorem 10 (Euler's Formula)

$$3p - 3q + 3r = 6$$
$$3p - q = 6q \qquad = 3p - 6$$

and hence by Theorem 13 Γ is maximal planar. □

Lemma 59 *If Γ is a minimal counter example to the four colour theorem then any minimal disconnecting set induces (at least) a cycle.*

Proof. Let S be a minimal disconnecting set that does not induce a cycle, then Γ can be drawn in the plane so that S connects two components: L which lies at the left of S, and R which lies to the right of S.

Choose a vertex v which is on the boundary of the 'exterior' region of L, and a vertex w in R with the same property. Then the edge $\{v, w\}$ can be added to Γ without violating planarity. However, Γ is a minimal counter example to the four colour theorem and so is a triangulation. But then by Lemma 58 Γ is maximal planar and hence this is a contradiction. Hence S must induce a cycle. □

Theorem 60 *If Γ is a minimal counter-example to the four colour theorem then it is 5-connected.*

Proof. Suppose that Γ is not 5-connected then by Lemma 59 there is a disconnecting triangle or square. In the case of a triangle the graphs on the inside and outside of the triangle can be 4-coloured by induction, and the colourings at the three vertices of the triangle can be matched up.

In the case of a square, label the vertices w, x, y, z then Γ may be split into two smaller graphs as follows:

- Let Γ_1 be the disconnecting ring together with the interior of the ring, and a new edge $\{w, y\}$ (on the outside of the ring).
- Let Γ_2 be the disconnecting ring together with the exterior of the ring, and a new edge $\{w, y\}$ (on the inside of the ring).

These two graphs are shown in Figure 9, and are again triangulations. By induction they can be 4-coloured.

A permutation of the colours in Γ_1 and Γ_2 so that w, x, y, z match up produces a 4-colouring of Γ, unless x and z have the same colour in one of Γ_1 and Γ_2 and different colours in the other.

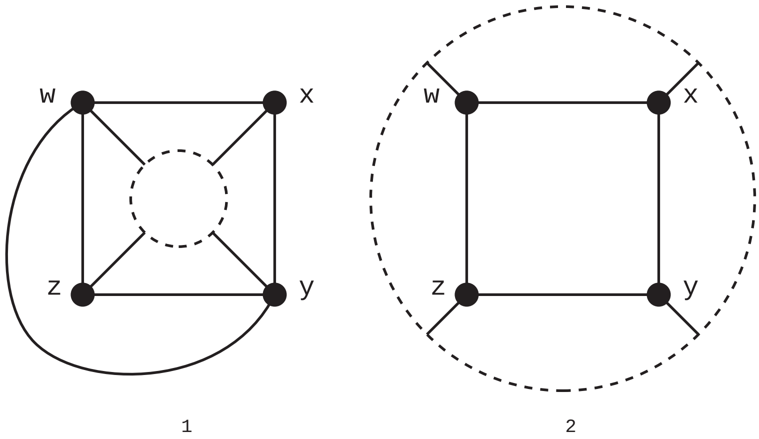

Figure 9: Two new graphs created from a graph with a disconnecting ring of length 4.

Suppose that x and z have the same colour in Γ_1 and different colours in Γ_2. If there is no Kempe chain from x to z in Γ_2 then the colour of x can be changed to be the same as that of z, and so a permutation of colours allows a matching to be made.

If there is a Kempe chain from x to z in Γ_2 then there can be no w-y chain and so w and y may be re-coloured with the same colour. Now in Γ_1 replace the edge $\{w, y\}$ with $\{x, z\}$ and re-colour. If w is the same colour as y then this colouring matches the new colouring of Γ_2. If w is the same colour as y then the original colouring of Γ_2 is used. □

Corollary 61 *A graph with a separating circuit of length less than 5 is reducible.*

The Birkhoff Diamond

Certain kinds of configuration are reducible, and they are classified accordingly. Those which can be proven reducible by traditional Kempe chain arguments are called D-reducible. An alternative approach to proving reducibility is to replace the interior of a bounding ring with something smaller: the induction argument is still valid but there are restrictions on how the vertices of the bounding ring are coloured. Configurations requiring this kind of argument are called C-reducible.

The "Birkhoff Diamond" is an example of a C-reducible configuration. It consists of a vertex of degree 5 that has 3 neighbours of degree 5, note that the degrees of its other neighbours are not known.

Theorem 62 *The Birkhoff Diamond configuration—a vertex of degree 5 with 3 neighbours of degree 5—is reducible.*

Proof. The configuration is shown in Figure 10 where a filled circle denoted a vertex of degree 5 and an unfilled circle denotes a vertex of unknown degree.

Disregard the inside of the hexagon, identify v_2 and v_4, and join v_6 to v_2 as shown in Figure 10. Note that if v_2 was adjacent to v_4 (shown by the dotted line in Figure 10) then v_3 would have degree 3 but by assumption all vertices have degree at least 5.

By induction the collapsed hexagon can be 4-coloured, and a simple case analysis reveals that there are 6 possibilities for this. These are shown in Figure 11 and the remainder of the proof is to deduce how to colour the interior of the hexagon.

In all but case 3 in Figure 11 the colouring is trivial. For case 3:

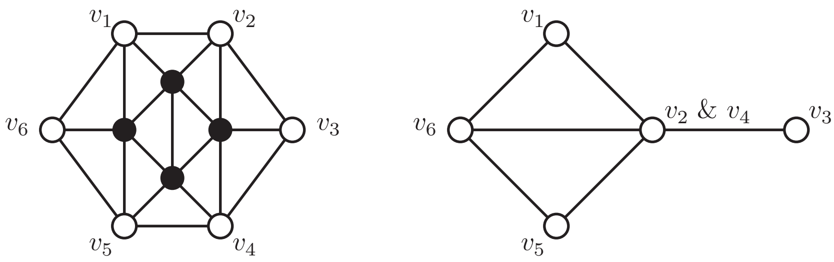

Figure 10: The Birkhoff Diamond and the collapsed hexagon.
A filled circle denoted a vertex of degree 5 and an unfilled circle denotes a vertex of unknown degree. If the dotted line was a vertex then v_3 would have degree 3—a contradiction.

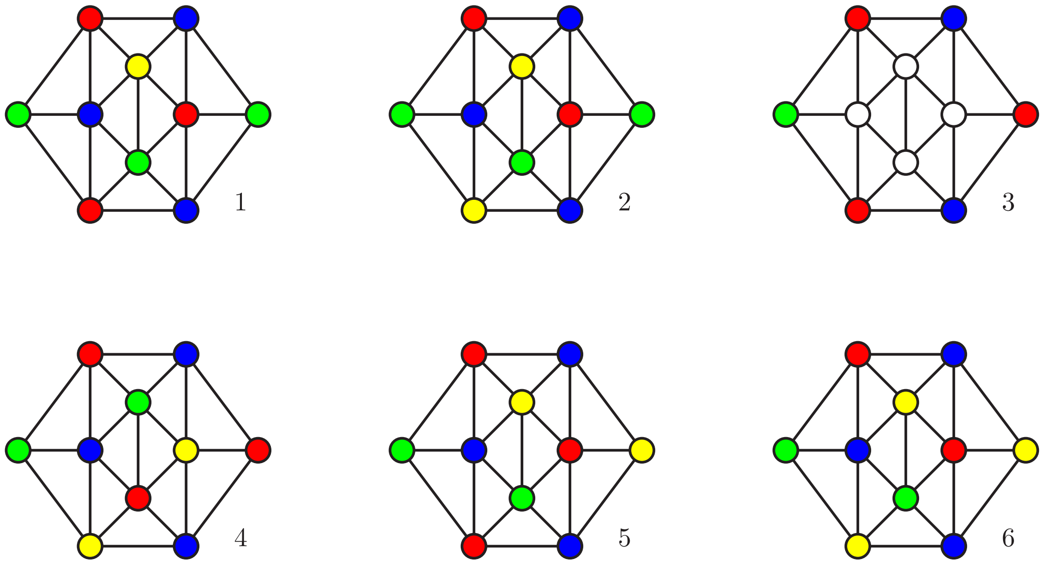

Figure 11: Possible colourings of the Birkhoff Diamond.

- If v_1, v_3, and v_5 are in the same red-yellow Kempe chain then v_4 can be re-coloured green and the colouring completed as shown in Figure 12.
- If this is not so, then either v_1 or v_5 is in a different red-yellow Kempe chain to v_3 and v_5 or v_1 (respectively) and so can be re-coloured yellow to give case 4.

In any case the original graph can be coloured, and so the Birkhoff Diamond is reducible. □

Other Reducible Configurations

Theorem 63 (Birkhoff) *If a graph has a separating circuit of length 5 such that both the interior and the exterior contain at least 2 vertices then the graph is reducible.*

Proof. Omitted. □

Of course as well as conditions for reducibility, conditions for irreducibility may be examined.

Theorem 64 (Wernicke) *In any minimal counter example to the four colour theorem there is a vertex of degree 5 which is adjacent to a vertex of degree 5 or 6.*

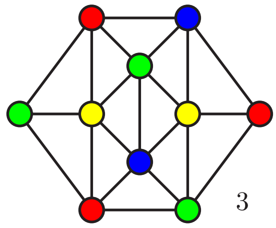

Figure 12: Re-colouring the Birkhoff Diamond to obtain reducibility.

Proof. Consider a triangulated graph that has no vertices of degree less than 5, and where no vertex of degree 5 is adjacent to a vertex of degree 5 or 6. Consider the regions of the graph which have a vertex of degree 5 or 6 on their boundary.

- If a region has a vertex of degree 5 on its boundary then the other two vertices cannot be of degree 5 or 6.
- If a region has a vertex of degree 6 on its boundary, then the other two vertices must have degree at least 6.

Hence there are at least $5p_5 + 2p_6$ such regions. Say there are r such regions then

$$\begin{aligned} r &\geqslant 5p_5 + 2p_6 \\ &\geqslant 5p_5 + 2p_6 - p_7 - 4p_8 - \cdots - (20-3i)p_i - \dots \\ &= \sum_{i=5}^{\infty} (20-3i)p_i \\ &= 20p - 3\sum_{i=5}^{\infty} ip_i \\ &= 20p - 10\sum_{i=5}^{\infty} ip_i + 7\sum_{i=5}^{\infty} ip_i \end{aligned}$$

Now by Lemma 11 (Handshaking) in a triangulated graph $\sum_{i=1}^{\infty} ip_i = 2q = 3r$ and since there are no vertices of degree less than 5 this gives

$$\begin{aligned} r &\geqslant 20p - 20q + 21r \\ &\geqslant 20(p - q + r) + r \\ &\geqslant 40 + r \end{aligned}$$

which is clearly a contradiction and so completes the proof. □

Using A Computer To Prove Reducibility

Clearly a computer can produce all possible colourings of the outer ring of a configuration. For 4 colours there are 3 ways to choose 2 pairs yielding finitely many possible colour changes, all of which can be tried by a computer. Thus the D-reducibility of a given configuration can be readily verified by computer.

Verifying C-reducibility is somewhat more difficult since it is infeasible to check all possibilities for identifying vertices on the ring or adding new vertices. It is preferable instead to spend time on producing a 'nice' unavoidable set of configurations.

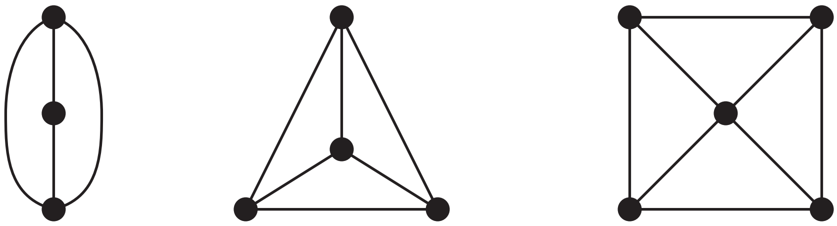

Figure 13: Configurations contained in all unavoidable sets.

(35.5.3) Discharging

Discharging is the process which yields an unavoidable set, and prudent choice of discharging algorithm will yield a 'nice' unavoidable set.

By Theorem 10 (Euler's Formula) and Lemma 11 (Handshaking), for a triangulation

$$\begin{aligned} q &= 3p - 6 \\ 2q &= \sum_{v \in V} \deg v \\ 12 &= \sum_{v \in V} 6 - \deg v \end{aligned}$$

The only positive contributions to this sum are from vertices of degree 5. Associate with each vertex a 'charge' of $6 - \deg v$ this charge is then dissipated according to some discharging algorithm so that where positive charges remain there are 'lots' of vertices of degree 5.

Any unavoidable set must contain vertices of degree 2, 3, and 4 as per Figure 13. Kempe's unavoidable set was completed by adding to this a vertex of degree 5, but as his fallacious proof shows, these configurations are not all reducible. The configurations of Figure 13 are omitted for clarity. Vertices of degree 5 are denoted as filled circles, vertices of degree 6 are denoted as unfilled circles, and vertices of other degrees are appropriately labelled. Hence Kempe's unavoidable set is denoted simply as $\{\bullet\}$.

Theorem 64 suggests another unavoidable set to be the vertices of degree 2, 3, and 4, together with two adjacent vertices of degree 5, and a vertex of degree 5 adjacent to a vertex of degree 6. This unavoidable set is denoted as

Kempe's unavoidable set is obtained from the discharging algorithm that does nothing: The unavoidable sets are those which, when the discharging algorithm has finished, have positive charge. Wernicke's unavoidable set may be produced from the following algorithm.

Algorithm 65 *Every vertex of degree 5 gives a charge of $\frac{1}{5}$ to each adjacent vertex that has degree at least 7.*

Theorem 66 *Algorithm 65 produces Wernicke's unavoidable set of configurations.*

Proof. It is clear that Wernicke's set of unavoidable configurations cannot be discharged by Algorithm 65. Conversely, suppose that a graph Γ has neither of these configurations then when Algorithm 65 has been applied all vertices of degree 5 and 6 will have no charge. Let v be a vertex of degree $k \geqslant 7$. Since Γ is triangulated v can be adjacent to at most $\frac{k}{2}$ vertices of degree 5 and thus have residual charge of at most

$$(6-k) + \frac{1}{5}\frac{k}{2} = 6 - \frac{9}{10}k \leqslant 6 - \frac{9}{10}7 < 0$$

which is clearly not equal to 12, the original charge, and so is a contradiction. □

Finally, an alternative algorithm.

Algorithm 67 *Each vertex of degree 5 gives a charge of $\frac{1}{3}$ to each adjacent vertex that has degree at least 7.*

Theorem 68 *The set *** is unavoidable.*

Proof. Suppose that Γ (is a triangulated graph with minimum vertex degree 5 and) contains none of the configurations in $\mathcal{U}$. Therefore each vertex of degree 5 is adjacent to at least 3 vertices of degree at least 7 and so is discharged by Algorithm 67.

Vertices of degree 6 are uneffected by Algorithm 67 and so finish with no charge.

Vertices of degree 7 are adjacent to at most 3 vertices of degree 5 and so receive a maximum charge of $-1 + 3\frac{1}{3} = 0$.

Vertices of degree $k \geqslant 8$ are adjacent to at most $\frac{k}{2}$ vertices of degree 5 and so receive a maximum charge of $(6-k) + \frac{1}{3}\frac{k}{2} = 6 + \frac{5}{6}k \leqslant 6 + \frac{5}{6}8 < 0$ and hence after applying Algorithm 67 the sum of charges in Γ is negative. Since the charge was originally 12 this is a contradiction. $\square$